Risk-Managing Bermudan Swaptions in a LIBOR Model

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This article presents a new approach to calculating swap vega per bucket in a LIBOR model. It shows that for some forms of volatility an approach based on recalibration may make estimated swap vega very uncertain, as the instantaneous volatility structure may be distorted by recalibration. This does not happen in the case of constant swap rate volatility.

An alternative approach not based on recalibration comes out of comparison with the swap market model. It accurately estimates vegas for any volatility function in few simulation paths. The key to the method is that the perturbation in LIBOR volatility is distributed in a clear, stable, and well-understood fashion, while in the recalibration method the change in volatility is hidden and potentially unstable.

The LIBOR interest rate model developed by Brace, Gatarek, and Musiela [1997], Jamshidian [1997], and Miltersen, Sandmann, and Sondermann [1997] is popular among both academics and practitioners alike. We will call this the BGM model.

One reason the LIBOR BGM model is popular is that it can risk-manage interest rate derivatives that depend on both the cap and swaption markets, which would make it a *central interest rate model.* It features lognormal LIBOR and almost lognormal swap rates, and thus also the market-standard Black formula for caps and swaptions. Approximate swaption volatility formulas such as in Hull and White [2000] have been shown to be of high quality (see Brace, Dunn, and Barton [1998]).

There remain a number of issues to be resolved to use BGM as a central interest rate model. One issue is the calculation of swap vega. A common and usually very successful method for calculating a Greek in a model equipped with a calibration algorithm is to perturb market input, recalibrate, and then revalue the option. The difference in value divided by the perturbation size is then an estimate for the Greek.

If this technique is applied to the calculation of swap vega in the LIBOR BGM model, however, it may (depending on the volatility function) yield estimates with high uncertainty. In other words, the standard error of the vega is relatively high. The uncertainty disappears, of course, if we increase the number of simulation paths, but the number required for clarity can far exceed 10,000, which is probably the maximum in a practical environment.

For a constant-volatility calibration, however, the vega is estimated with low uncertainty. The number of simulation paths needed for clarity of vega thus depends on the chosen calibration. The reason is that for certain calibrations, under a perturbation, the additional volatility is distributed unevenly and one might even say unstably over time. For a constant-volatility calibration, of course, this additional volatility is naturally distributed evenly over time. It follows that there is higher correlation between the discounted payoffs along the original path and perturbed volatility. As the vega is the expectation of the difference between these payoffs (divided by the perturbation size), the standard error will be lower.

We develop a method that is not based on recalibration to compute swap vega per bucket in the LIBOR BGM model. It may be used to calculate swap vega in the presence of any volatility function, with predictability at 10,000 or fewer simulation paths. The strength of the method is that it accurately estimates swap vegas for any volatility function and in few simulation paths.

The key to the method is that the perturbation in the LIBOR volatility is distributed in a clear, stable, and well-understood fashion, while in the recalibration method the change in volatility is hidden and potentially unstable. The method is based on keeping swap rate correlation fixed but increasing the instantaneous volatility of a single swap rate evenly over time, while all other swap rate volatilities remain unaltered.

It is important to verify that a calculation method reproduces the correct numbers when the answer is known. We benchmark our swap vega calculation method using Bermudan swaptions for two reasons. First, a Bermudan swaption is a complicated enough (swap-based) product (in a LIBOR-based model) that depends non-trivially on the swap rate volatility dynamics; for example, its value depends also on swap rate correlation. Second, a Bermudan swaption is not as complicated as some other more exotic interest rate derivatives, and some intuition exists about its vega behavior. We show for Bermudan swaptions that our method yields almost the same swap vega as found in a swap market model.

Glasserman and Zhao [1999] provide efficient algorithms for calculating risk sensitivities, given a perturbation of LIBOR volatility. Our problem differs from theirs in that we derive a method to calculate the perturbation of LIBOR volatility to obtain the correct swap rate volatility perturbation for swaption vega. The Glasserman and Zhao approach may then be applied to efficiently compute the swaption vega, with the LIBOR volatility perturbation we find using our method.

I. RECALIBRATION APPROACH

We first consider examples of the recalibration approach to computing swap vega. Three calibration methods are considered. We show that, for two of the three methods, the resulting vega is hard to estimate and many simulation paths are needed for clarity.

The notation is as follows. A BGM model features a tenor structure $0 < T_1 < \ldots < T_{N+1}$ and N forward rates L_i accruing from T_i to T_{i+1}, $i = 1, \ldots, N$. Each forward rate is modeled as a geometric Brownian motion under its forward measure:

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = \bar{\sigma}_i(t) \cdot \mathrm{d}\bar{W}^{i+1}(t) \text{ for } 0 \le t \le T_i$$

The positive integer d is referred to as the *number of factors* of the model. The function σ_i: $[0, T_i] \to \mathbb{R}^d$ is the volatility vector function of the i-th forward rate. The k-th component of this vector corresponds to the k-th Wiener factor of the Brownian motion. $\bar{W}^{i+1}$ is a d-dimensional Brownian motion under the forward measure $\mathbb{Q}_{i+1}$.

A discount bond pays one unit of currency at maturity. The time t price of a discount bond with maturity T_i is denoted by $B_i(t)$. The forward rates are related to discount bond prices as follows:

$$L_i(t) = \frac{1}{\delta_i}\left\{ \frac{B_i(t)}{B_{i+1}(t)} - 1 \right\}$$

where δ_i is the accrual factor for the time span $[T_i, T_{i+1}]$.

The swap rate corresponding to a swap starting at T_i and ending at T_{j+1} is denoted by $S_{i:j}$. The swap rate is related to discount bond prices as follows:

$$S_{i:j}(t) = \frac{B_i(t) - B_{j+1}(t)}{\mathrm{PVBP}_{i:j}(t)}$$

where PVBP denotes the *present value of a basis point*:

$$\mathrm{PVBP}_{i:j}(t) = \sum_{k=i}^{j} \delta_k B_{k+1}(t)$$

It is understood that $\mathrm{PVBP}_{i:j} \equiv 0$ whenever $j < i$.

We consider the swap rates $S_{1:N}, \ldots, S_{N:N}$ corresponding to the swaps underlying a coterminal Bermudan swaption.[1] Swap rate $S_{i:N}$ is a martingale under its forward swap measure $\mathbb{Q}_{i:N}$. We may thus implicitly define its volatility vector $\bar{\sigma}_{i:N}$ by:

$$\frac{\mathrm{d}S_{i:N}(t)}{S_{i:N}(t)} = \bar{\sigma}_{i:N}(t) \cdot \mathrm{d}\bar{W}^{i:N}(t) \text{ for } 0 \le t \le T_i \quad (1)$$

In general, $\bar{\sigma}_{i:N}$ will be stochastic because swap rates are not lognormally distributed in the BGM model, although they are very close to lognormal as shown, for

EXHIBIT 1

Market European Swaption Volatilities

Expiry (Y)	1	2	3	...	28	29	30
Tenor (Y)	30	29	28	...	3	2	1
Swaption Volatility	15.0%	15.2%	15.4%	...	20.4%	20.6%	20.8%

example, by Brace, Dunn, and Barton [1998]. Because of near lognormality, the Black formula approximately holds for European swaptions. There are closed-form formulas for the swaption's Black implied volatility; see, for example, Hull and White [2000].

We model LIBOR instantaneous volatility as constant in between tenor dates (piecewise-constant). A volatility structure $\{\bar{\sigma}_i(\cdot)\}_{i=1}^{N}$ is *piecewise-constant* if:

$$\bar{\sigma}_i(t) = (\text{const}), \quad t \in [T_{i-1}, T_i)$$

The volatility will sometimes be modeled as time-homogeneous. To define this, first define a fixing to be one of the time points $T_1, \ldots, T_N$. Define ι: $[0, T] \to \{1, \ldots, N\}$:

$$\iota(t) = \#\{ \text{ fixings in } [0, t) \}$$

A volatility structure is said to be time-homogeneous if it depends only on the index to maturity $i - \iota(t)$.

Three volatility calibration methods are considered:

1. (THFRV)—*Time-homogeneous forward rate volatility.* This approach is based on ideas of Rebonato [2001]. Because of the time-homogeneity restriction, there are as many parameters as market swaption volatilities. A Newton-Rhapson sort of solver may be used to find the exact calibration solution (if there is one).
2. (THSRV)—*Time-homogeneous swap rate volatility.* The algorithm for calibrating with such a volatility function is a two-stage bootstrap. The first and the second stage are described in Equation (6.20) and Section 7.4 of Brigo and Mercurio [2001].
3. (CONST)—*Constant forward rate volatility.* Note that constant forward rate volatility implies constant swap rate volatility. The corresponding calibration algorithm is similar to the second stage of the two-stage bootstrap.

All calibration methods have in common that the forward rate correlation structure is calibrated to a historical correlation matrix using principal components analysis (PCA); see Hull and White [2000]. Correlation is assumed to evolve time-homogeneously over time.

We consider a 31NC1 coterminal Bermudan payer's swaption deal struck at 5% with annual compounding. The notation xNCy denotes an "x non-call y" Bermudan option, which is exercisable as a swap with a maturity of x years from today but is callable only after y years. The option is callable annually.

The BGM tenor structure is $0 < 1 < 2 < \cdots < 31$. All forward rates are taken to equal 5%. The time zero forward rate instantaneous correlation is assumed following Rebonato [1998, p. 63] as:

$$\rho_{ij}(0) = e^{-\beta|T_i - T_j|}$$

where β is chosen to equal 5%. The market European swaption volatilities were taken as displayed in Exhibit 1.

To determine the exercise boundary, we use the Longstaff and Schwartz [2001] least squares Monte Carlo method. Only a single explanatory variable is considered, namely, the swap net present value (NPV). Two regression functions are employed, a constant and a linear term.

For each bucket a perturbation $\Delta\sigma(\approx 10^{-8})$ is applied to the swaption volatility in the calibration input data.[2] The model is recalibrated, and we check to see that the calibration error for all swaption volatilities is a factor 10^6 lower than the volatility perturbation. The Bermudan swaption is repriced through Monte Carlo simulation using the exact same random numbers.

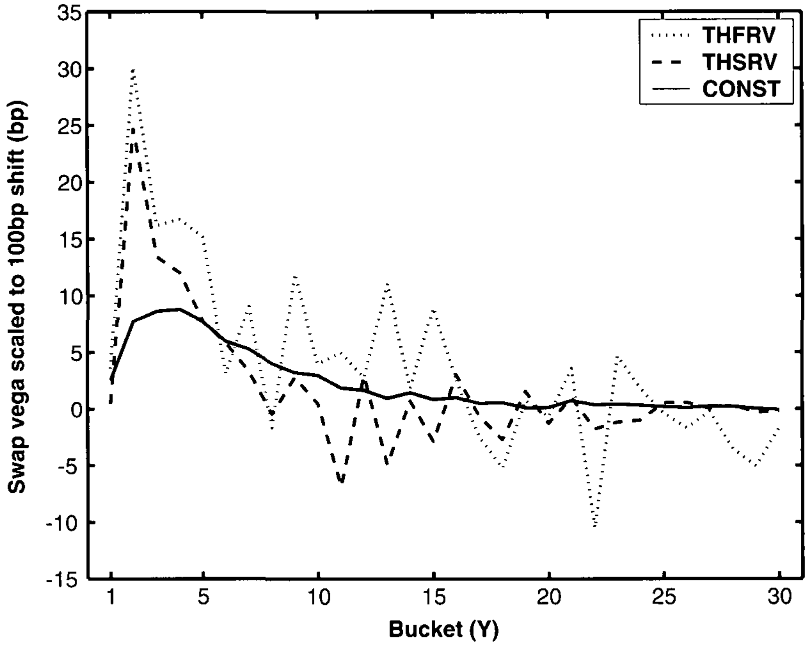
Denote the original price by V and the perturbed price by $V_{i:N}$. Then the recalibration method of estimating swap vega $\mathcal{V}_{i:N}$ for bucket i is given by:

$$\mathcal{V}_{i:N} = \frac{V_{i:N} - V}{\Delta\sigma} \tag{2}$$

Usually the swap vega is denoted in terms of a shift in the swaption volatility. For example, consider a 100 basis point (bp) shift in the swaption volatility. The swap vega scaled to a 100 bp shift $V_{i:N}^{100\text{bp}}$ is then defined by

EXHIBIT 2

Recalibration Swap Vega Results for 10,000 Simulation Paths



$$\mathcal{V}_{i:N}^{100\text{bp}} = (0.01) \cdot \mathcal{V}_{i:N}$$

Swap vega results for a Monte Carlo simulation of 10,000 scenarios are displayed in Exhibit 2. The standard errors (SEs) are displayed separately in Exhibit 3. The levels of SE for THFRV and CONST are 6.00 and 0.25, respectively. The number of paths needed for THFRV to obtain the same SE as CONST is thus $(6/0.25)^2 \times 10{,}000 = 5.8\text{M}$. For THSRV, we find 1.4M paths are needed.

Exhibit 4 displays the THFRV vega for 1 million simulation paths.

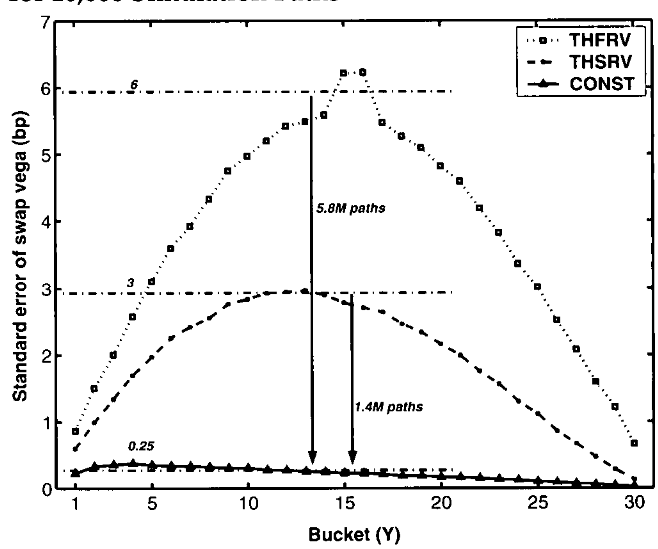
II. EXPLANATION

The key to explanation of the vega results under recalibration is the change in swap rate instantaneous variance after recalibration. For the THFRV and THSRV recalibration approaches, the instantaneous variance increment (in the limit) is completely different from a constant- volatility increment. This holds for all buckets.

For illustration, we consider the volatility perturbation shown in Exhibit 5. For THFRV, the distribution of the variance increment is concentrated in the beginning and ending time periods, and is even negative in the second time period. This is at variance with the natural and intuitive even distribution in the CONST recalibration.

EXHIBIT 3

Empirical Standard Errors of Vega for 10,000 Simulation Paths



From Equation (2), it follows that the simulation variance of the vega is given by

$$\begin{aligned}\mathrm{Var}\left[\mathcal{V}_{i:N}^{100\text{bp}}\right] &= c^2 \mathrm{Var}\left[P_{i:N} - P\right] \\ &= c^2 \left\{ \mathrm{Var}\left[P_{i:N}\right] - 2\mathrm{Cov}\left[P_{i:N}, P\right] + \mathrm{Var}\left[P\right] \right\}\end{aligned} \tag{3}$$

EXHIBIT 4

Recalibration THFRV Vega Results for 1 Million Simulation Paths

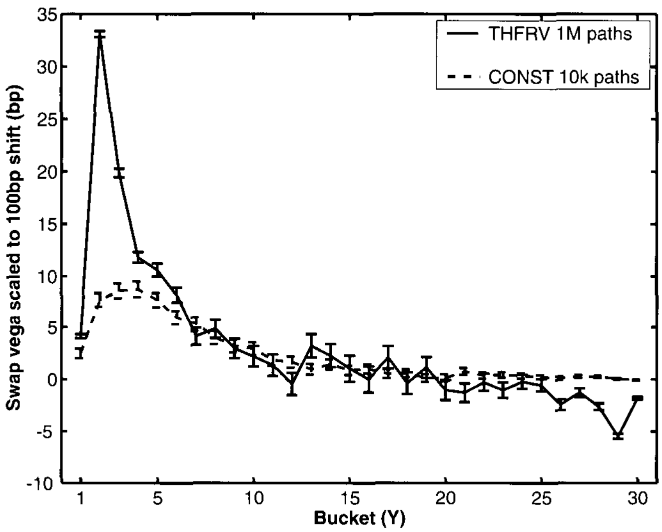
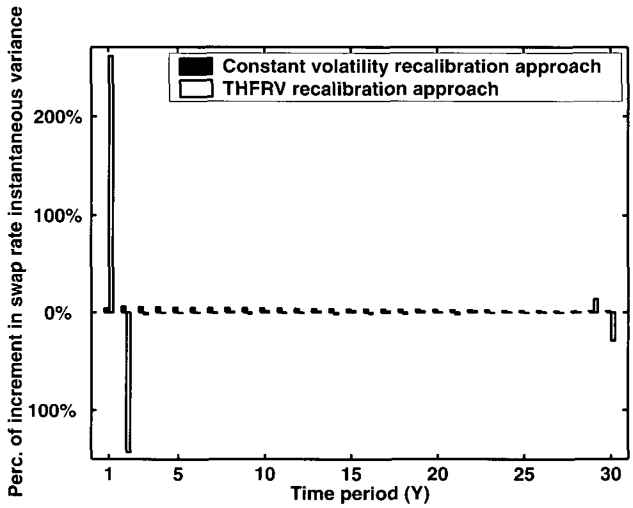


EXHIBIT 5

Observed Change in Swap Rate Instantaneous Variance for THFRV and CONST Recalibration Approach



where P and $P_{i:N}$ are the payoffs along the path of the original and the perturbed model, respectively. Here $c := 0.01/\sigma_{i:N}$.

The vega standard error is thus minimized if there is high covariance between the discounted payoffs in the original and the perturbed model. This does *not* occur for a perturbation such as dictated by THFRV, because the stochasticity in the simulation is basically moved around to other time periods (in our case from period 2 to period 1). Because the rate increments over different time periods are *independent*, this leads to a reduced covariance, leading in turn to a higher standard error of the vega.

There is higher covariance between the payoffs under the perturbations of variance implied by the CONST calibration, because then each independent time period maintains approximately the same level of variance; no stochasticity is moved to other random sources. From Equation (3), it then follows that the standard error is lower.

III. SWAP VEGA AND THE SWAP MARKET MODEL

An alternative method for calculating swap vega has the advantage that the estimates of vega have a low standard error for any volatility function. The first step is to study the definition of swap vega in the swap market model, which we will extend to the LIBOR BGM model. This will give us an alternative method to calculate swap vega per bucket.

How much our dynamically managed hedging portfolio should hold in European swaptions is essentially determined by the swap vega per bucket. The latter is the derivative of the exotic price with respect to the Black implied swaption volatility.

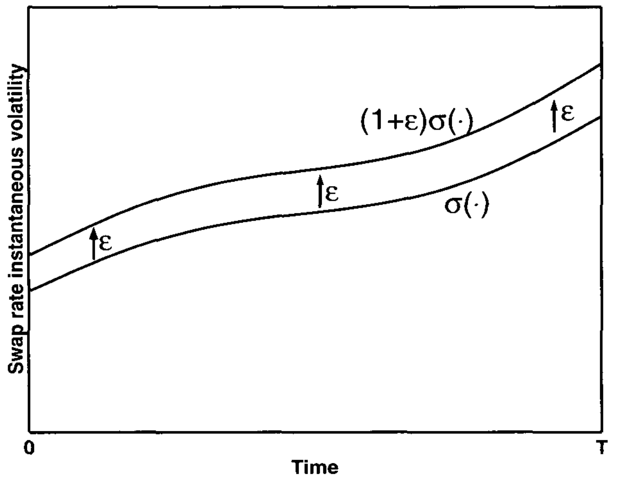
Consider a swap market model $\mathcal{S}$. In the model, swap rates are lognormally distributed under their forward swap measure. This means that all swap rate volatility functions $\overline{\sigma}_{i:N}(\cdot)$ of Equation (1) are deterministic. The Black implied swaption volatility $\sigma_{k:N}$ is given by

$$\sigma_{k:N} = \sqrt{\frac{1}{T_k}\int_0^{T_k} |\bar{\sigma}_{k:N}(s)|^2 \mathrm{d}s}$$

As may be seen in this equation, there are an uncountable number of perturbations of the swap rate instantaneous volatility that produce the same perturbation as the Black implied swaption volatility. There is, however, a natural one-dimensional parameterized perturbation of the swap rate instantaneous volatility, namely, a simple proportional increment. This is illustrated in Exhibit 6.

We define swap vega in the swap market model as follows. Denote the price of an interest rate derivative in a swap market model $\mathcal{S}$ by V. Consider a perturbation of the swap rate instantaneous volatility given by

EXHIBIT 6
Natural Increment of Black Implied Swaption Volatility



$$\bar{\sigma}^{\varepsilon}_{k:N}(\cdot) = (1+\varepsilon)\,\bar{\sigma}_{k:N}(\cdot) \tag{4}$$

where the shift applies only to k:N. Denote the corresponding swap market model by $\mathcal{S}_{k:N}(\varepsilon)$. Note that the implied swaption volatility in $\mathcal{S}_{k:N}(\varepsilon)$ is given by $\sigma^{\varepsilon}_{k:N} = (1 + \varepsilon)\sigma_{k:N}$. Denote the price of the derivative in $\mathcal{S}_{k:N}(\varepsilon)$ by $V_{k:N}(\varepsilon)$. Then the swap vega per bucket $\mathcal{V}_{k:N}$ is defined as

$$\mathcal{V}_{k:N} = \lim_{\varepsilon \to 0} \frac{V_{k:N}(\varepsilon) - V}{\varepsilon \sigma_{k:N}} \tag{5}$$

Equation (5) is the derivative of the exotic price with respect to the Black implied swaption volatility. In conventional notation we may write

$$\begin{aligned}\mathcal{V}_{k:N} &= \frac{\partial V}{\partial \sigma_{k:N}} \\ &= \lim_{\Delta\sigma_{k:N} \to 0} \frac{V(\sigma_{k:N} + \Delta\sigma_{k:N}) - V(\sigma_{k:N})}{\Delta\sigma_{k:N}}\end{aligned} \tag{6}$$

In Equation (5) $\varepsilon\sigma_{k:N}$ is equal to the swaption volatility perturbation $\Delta\sigma_{k:N}$, and $V_{k:N}(\varepsilon)$ and V denote the prices of the derivative in models where the k-th swaption volatility equals $\sigma_{k:N} + \Delta\sigma_{k:N}$ and $\sigma_{k:N}$, respectively.

The swap rate volatility perturbation in Equation (4) defines a relative shift. It is also possible to apply an absolute shift in the form of

$$\bar{\sigma}^{\varepsilon}_{k:N}(\cdot) = \left(1 + \frac{\varepsilon}{|\bar{\sigma}_{k:N}(\cdot)|}\right)\bar{\sigma}_{k:N}(\cdot) \tag{7}$$

where the shift applies only to $k : N$. This ensures that the absolute level of the swap rate instantaneous volatility is increased by an amount ε. Note that the relative and absolute perturbation are equivalent when the instantaneous volatility is constant over time.

The method for calculating swap vega per bucket is largely the same for both relative and absolute perturbation (but we will point out any differences). The first difference is in the change in swaption implied volatility $\Delta\sigma_{k:N}$ of Equation (6); namely, straightforward calculations reveal that the perturbed volatility satisfies

$$\sigma^{\varepsilon}_{k:N} = \sigma_{k:N} + \varepsilon\,\frac{\frac{1}{T_k}\int_0^{T_k} |\bar{\sigma}_{k:N}(s)|\mathrm{d}s}{\sigma_{k:N}} + \mathcal{O}(\varepsilon^2)$$

IV. ALTERNATIVE METHOD FOR CALCULATING SWAP VEGA

An alternative method for calculating swap vega in the BGM framework may be applied to any volatility function to yield accurate vega with a small number of simulation paths. The method is based on a perturbation in the forward rate volatility to match a constant swap rate volatility increment. Rebonato [2002] also derives this method in terms of covariance matrices, but our derivation is explicitly in terms of volatility vectors.

Swap rates are not lognormally distributed in the LIBOR BGM model. This means that swap rate instantaneous volatility is stochastic. The stochasticity is almost invisible as shown empirically, for example, by Brace, Dunn, and Barton [1998]. D'Aspremont [2002] shows that the swap rate is uniformly close to a lognormal martingale.

Hull and White [2000] show that the swap rate volatility vector is a weighted average of forward LIBOR volatility vectors:

$$\begin{aligned}\bar{\sigma}_{i:N}(t) &= \sum_{j=1}^{N} w_j^{i:N}(t)\bar{\sigma}_j(t) \\ w_j^{i:N}(t) &= \frac{\delta_j \gamma_j^{i:N}(t) L_j(t)}{1+\delta_j L_j(t)} \\ \gamma_j^{i:N}(t) &= \frac{B_i(t)}{B_i(t) - B_{N+1}(t)} - \\ &\quad \frac{\mathrm{PVBP}_{i:(j-1)}(t)}{\mathrm{PVBP}_{i:N}(t)}\end{aligned} \tag{8}$$

where the weights $w^{i:N}$ are in general state-dependent.

Hull and White derive an approximating formula for European swaption prices that is based on evaluating the weights in Equation (8) at time zero. This is a good approximation by virtue of the near lognormality of swap rates in the LIBOR BGM model. We denote the resulting swap rate instantaneous volatility by $\bar{\sigma}^{HW}_{i:N}$ as follows:

$$\bar{\sigma}^{\mathrm{HW}}_{i:N}(t) = \sum_{j=i}^{N} w^{i:N}_{j}(0)\bar{\sigma}_{j}(t) \tag{9}$$

When we write $\omega^{i:N}_{j} := \omega^{i:N}_{j}(0)$ and adopt the convention that

$$\bar{\sigma}_i(t) = \bar{\sigma}_{i:N}(t) = 0 \quad \text{when} \quad t > T_i$$

a useful form of Equation (9) is:

$$\begin{array}{lclcccc} \bar{\sigma}^{\mathrm{HW}}_{1:N}(t) & = & w^{1:N}_{1}\bar{\sigma}_1(t) & + & \dots & + & w^{1:N}_{N}\bar{\sigma}_N(t) \\ \vdots & & & & \ddots & & \vdots \\ \bar{\sigma}^{\mathrm{HW}}_{N:N}(t) & = & & & & & w^{N:N}_{N}\bar{\sigma}_N(t) \end{array} \tag{10}$$

If W is the upper triangular non-singular weight matrix (with upper triangular inverse W^{-1}), these volatility vectors can be jointly related through the matrix equation:

$$\left[\, \bar{\sigma}_{\cdot:N} \,\right] = W \left[\, \bar{\sigma}_{\cdot} \,\right]$$

The swap rate volatility under relative perturbation [Equation (4)] of the k-th volatility is

$$\left[\, \bar{\sigma}_{\cdot:N} \,\right] \rightarrow \left[\, \bar{\sigma}_{\cdot:N} \,\right] + \varepsilon \left[\, 0 \ \dots \ 0 \ \ \bar{\sigma}_{k:N} \ \ 0 \ \dots \ 0 \,\right]^{\mathrm{T}}$$

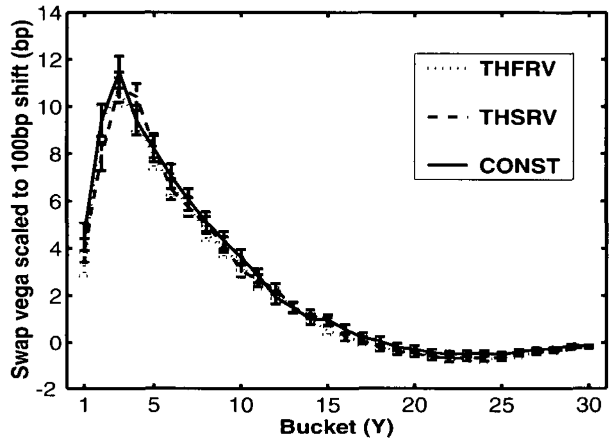
Note that the swap rate correlation is left unaltered. The corresponding perturbation in the BGM volatility vectors is given by

$$\left[\, \bar{\sigma}_{\cdot} \,\right] \rightarrow \left[\, \bar{\sigma}_{\cdot} \,\right] + \varepsilon W^{-1} \left[\, 0 \ \dots \ 0 \ \ \bar{\sigma}_{k:N} \ \ 0 \ \dots \ 0 \,\right]^{\mathrm{T}} \tag{11}$$

Note that only the volatility vectors $\bar{\sigma}_k(t), \dots, \bar{\sigma}_N(t)$ are affected (due to the upper triangular nature of W^{-1}), which are the vectors that underlie $\bar{\sigma}_{k:N}(t)$ in the Hull and White approximation. With the new LIBOR volatility vectors, prices can be recomputed in the BGM model and the vegas calculated.

EXHIBIT 7

Swap Vega Results for 10,000 Simulation Paths



Error bars denote 95% confidence bound based on the standard error.

V. NUMERICAL RESULTS

We demonstrate the algorithm in a simulation with 10,000 paths. The results are displayed in Exhibit 7. Note that the approach yields slightly negative vegas for buckets 17-30.

In the appendix we show that negative values are not a spurious result. That is, for the analytically tractable setup of a two-stock Bermudan option, negativity of vega occurs with correlation ≈ 1, and volatilities for short expiration dates are higher than volatilities at longer expiration dates—this of course is in a typical interest rate setting.

The vegas were also calculated for the absolute perturbation method in results not displayed. The differences in the vegas for the two methods are minimal; for any vega with absolute value above 1 bp, the difference is less than 4%, and for any vega with absolute value below 1 bp, the difference is always less than a third of a basis point.

VI. COMPARISON WITH THE SWAP MARKET MODEL

The swap market model (SMM) is the canonical model for computing swap vega per bucket. We compare the LIBOR BGM model and a swap market model with the very same swap rate quadratic cross-variation structure. Approximate equivalence between the two models has been established by Joshi and Theis [2002, Equation (3.8)].

EXHIBIT 8

Swap Vega per Bucket Test Results for Varying Strikes—10,000 Simulation Paths

BGM LIBOR MODEL

Fixed Rate	2%	3%	3.5%	4%	4.5%	5%	6%	7%	8%	9%	10%	12%	15%
Value	2171	1476	1138	829	585	410	210	112	64	36	21	8	2
	(4)	(5)	(5)	(5)	(5)	(4)	(3)	(2)	(2)	(1)	(1)	(1)	(0)
1Y	-2.0	-2.0	2.6	10.9	11.1	7.0	1.2	0.1	0.0	0.0	0.0	0.0	0.0
2Y	1.5	1.6	1.0	2.6	5.7	6.8	4.0	1.0	0.0	0.0	0.0	0.0	0.0
3Y	0.0	0.0	-0.3	0.1	2.5	4.5	4.1	2.1	1.0	0.3	0.0	0.0	0.0
4Y	0.0	0.0	-0.1	-0.1	1.1	2.7	4.4	3.6	2.0	1.1	0.5	0.2	0.1
5Y	0.0	0.0	-0.1	-0.2	0.4	1.5	3.7	3.6	2.7	1.5	1.0	0.3	0.1
6Y	0.0	0.0	-0.1	-0.2	0.1	0.8	2.1	2.5	2.0	1.7	1.2	0.3	0.2
7Y	0.0	0.0	-0.1	-0.2	0.0	0.3	1.3	1.8	1.8	1.6	1.1	0.5	0.0
8Y	0.0	0.0	0.0	-0.1	-0.1	0.1	0.7	1.3	1.5	1.3	1.3	0.9	0.3
9Y	0.0	0.0	0.0	-0.1	-0.1	0.0	0.3	0.7	0.8	0.8	0.8	0.6	0.3
10Y	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.3	0.3	0.4	0.4	0.3	0.2
Total Vega	-0.5	-0.4	2.9	12.8	20.8	23.8	21.9	16.9	12.3	8.8	6.2	3.1	1.0

SWAP MARKET MODEL

Fixed Rate	2%	3%	3.5%	4%	4.5%	5%	6%	7%	8%	9%	10%	12%	15%
Value	2172	1480	1146	841	592	411	204	109	61	34	19	7	1
	(6)	(6)	(6)	(5)	(5)	(4)	(4)	(3)	(2)	(1)	(1)	(1)	(0)
1Y	-1.9	-0.7	4.4	11.3	11.5	6.2	0.4	0.0	0.0	0.0	0.0	0.0	0.0
2Y	1.6	1.6	1.1	2.2	5.2	7.5	3.6	0.5	0.0	0.0	0.0	0.0	0.0
3Y	0.0	-0.1	-0.4	0.0	2.0	4.6	4.7	2.2	0.6	0.2	0.0	0.0	0.0
4Y	0.0	-0.1	-0.2	-0.1	0.9	2.7	4.8	3.7	1.7	0.8	0.3	0.1	0.0
5Y	0.0	0.0	-0.2	-0.2	0.4	1.6	3.7	3.0	2.3	1.2	0.5	0.1	0.0
6Y	0.0	0.0	-0.1	-0.2	0.1	0.8	2.6	3.3	3.1	2.3	1.2	0.2	0.0
7Y	0.0	0.0	-0.1	-0.2	-0.1	0.3	1.3	2.0	1.9	1.3	1.4	0.8	0.1
8Y	0.0	0.0	0.0	-0.1	-0.1	0.1	0.8	1.3	1.5	1.5	1.2	0.6	0.2
9Y	0.0	0.0	0.0	-0.1	-0.1	0.0	0.4	0.9	1.0	1.0	0.9	0.7	0.3
10Y	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.3	0.4	0.5	0.5	0.4	0.3
Total Vega	-0.3	0.6	4.5	12.6	19.9	23.8	22.3	17.2	12.5	8.8	6.0	2.9	0.9

Prices and vegas are in basis points. Standard errors in parentheses.

We perform the test for an 11NC1 pay-fixed Bermudan option on a swap with annual fixed and floating payments. A single-factor LIBOR BGM model is used with constant volatility calibrated to the euro cap volatility curve of October 10, 2001. The zero rates were taken to be flat at 5%. In the Monte Carlo simulation of the SMM we apply the discretization suggested in Lemma 5 of Glasserman and Zhao [2000].

Results appear in Exhibit 8, and are displayed partially in Exhibits 9 and 10. In this particular case, the BGM LIBOR model reproduces the swap vegas of the swap market model very accurately.

VII. CONCLUSIONS

We have presented a new approach to calculating swap vega per bucket in the LIBOR BGM model. We show that for some forms of the volatility an approach based on recalibration may lead to great uncertainty in estimated swap vega, as the instantaneous volatility structure may be distorted by recalibration. This does not happen in the case of constant swap rate volatility.

We derive an alternative approach that is not based on recalibration, using the swap market model. The method accurately estimates swaption vegas for any volatility func-

Exhibit 9
Comparison of LMM and SMM for Swap Vega per Bucket

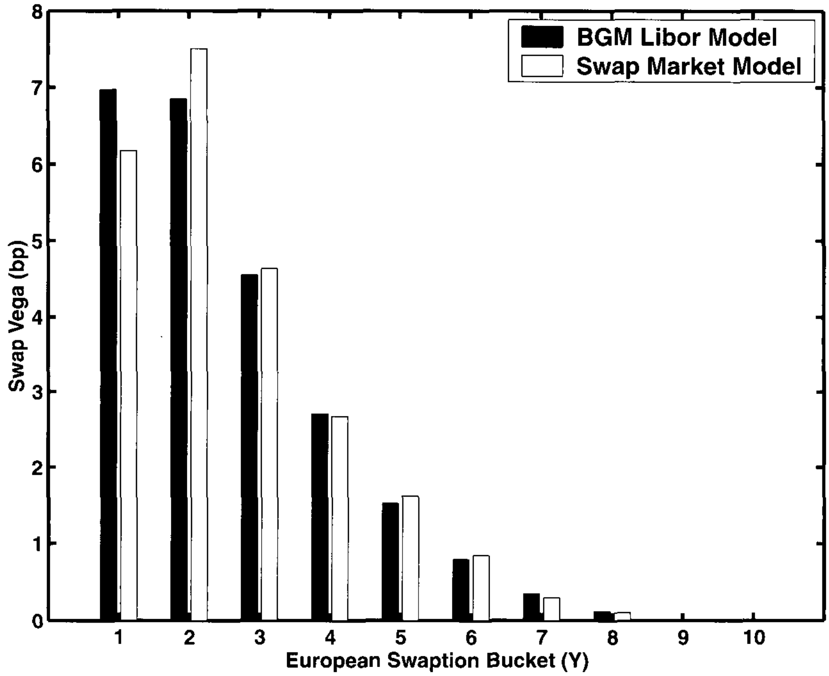
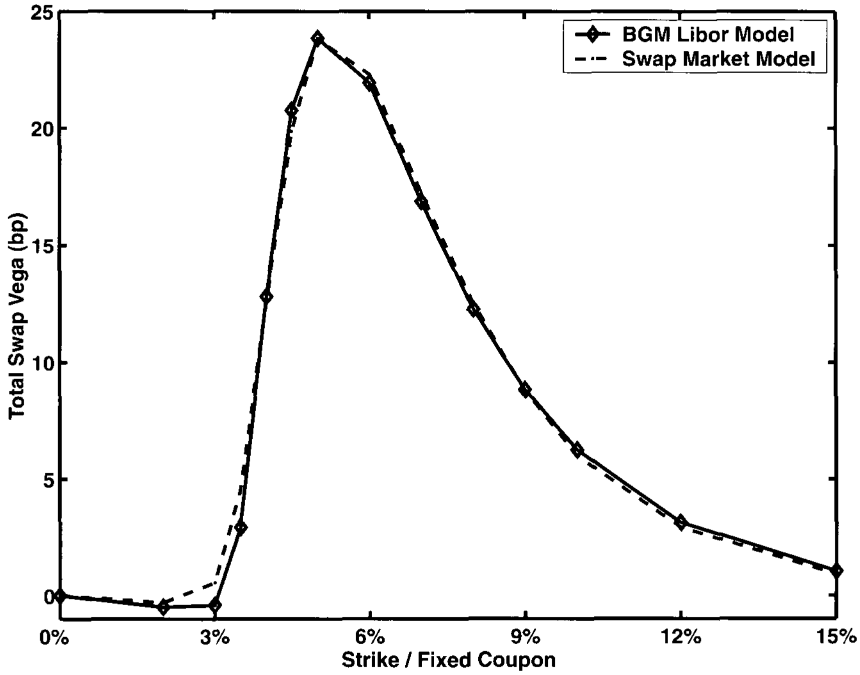


Exhibit 10
Comparison of LMM and SMM for Total Swap Vega Against Strike



tion and at a small number of simulation paths.

The key to the method is that the perturbation in the LIBOR volatility is distributed in a clear, stable, and well-understood fashion, but in the recalibration method the change in volatility is hidden and potentially unstable. We also show for a Bermudan swaption deal that our method yields almost the same swap vega as a swap market model.

APPENDIX

Negative Vega for a Two-Stock Bermudan Option

We examine a two-stock Bermudan option to show that its vega per bucket is negative in certain situations. The holder of a two-stock Bermudan option has the right to call the first stock S_1 at strike K_1 at time T_1; if the holder decides to hold the option, the right remains to call the second stock S_2 at strike K_2 at time T_2; if this right is not exercised, then the option becomes worthless. Here $T_1 < T_2$.

The Bermudan option is valued under standard Black-Scholes conditions. Under the risk-neutral measure, the stock prices satisfy the stochastic differential equations:

$$\frac{\mathrm{d}S_i}{S_i} = r\mathrm{d}t + \sigma_i \mathrm{d}W_i \text{ for } i = 1,2$$
$$\mathrm{d}W_1 \mathrm{d}W_2 = \rho \mathrm{d}t$$

where σ_i is the volatility of the i-th stock, and W_i, $i = 1, 2$, are Brownian motions under the risk-neutral measure, with correlation ρ. It follows that the time T_1 stock prices are distributed as follows:

$$S_i(T_1) = F(S_i(0), 0; T_1)$$
$$\exp\left\{ \sigma_i \sqrt{T_1} Z_i - \frac{1}{2}\sigma_i^2 T_1 \right\} \text{ for } i = 1,2 \tag{A-1}$$

where the pair (Z_1, Z_2) is standard bivariate normally distributed with correlation ρ and where

$$F(S,t;T) := S \exp\Big\{ r(T-t) \Big\} \tag{A-2}$$

is the time t forward price for delivery at time T of a stock with current price S.

At time T_1, the holder of the Bermudan option will choose whichever of two alternatives has a higher value: either calling the first stock, or holding the option on the second stock; the value of the latter is given by the Black-Scholes formula.

Therefore the (cash-settled) payoff $V(S_1(T_1), S_2(T_1), T_1)$ of the Bermudan at time T_1 is given by:

$$\max\Big\{ \big(S_1(T_1) - K_1 \big)_+, \text{BS}_2\big(S_2(T_1), T_1 \big) \Big\} \tag{A-3}$$

where BS is the Black-Scholes formula:

$$\begin{aligned} \text{BS}_i(S,T) &= e^{-r(T_i - T)}\{ F(S,T;T_i)N(d_1^{(i)}) - K_i N(d_2^{(i)}) \} \\ d_{1,2}^{(i)}(S,T) &= \frac{\ln\big(F(S,T;T_i)/K_i\big) \pm \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}} \end{aligned} \tag{A-4}$$

where $N(\cdot)$ is the cumulative normal distribution function.

The time zero value $V(S_1, S_2, 0)$ of the Bermudan option may thus be computed by a bivariate normal integration of the discounted version of the payoff in Equation (A-3):

$$V(S_1, S_2, 0) = e^{-rT_1}\, \mathbb{E}\Big[V\big(T_1, S_1(T_1), S_2(T_1) \big) \Big]$$

The vega per bucket $\mathcal{V}_i$ is defined as

$$\mathcal{V}_i := \frac{\partial V(S_1, S_2, 0)}{\partial \sigma_i} \text{ for } i = 1,2$$

The vega may be numerically approximated by finite differences:

$$\mathcal{V}_i = \frac{V(S_1, S_2, 0; \sigma_i + \Delta\sigma_i) - V(S_1, S_2, 0; \sigma_i)}{\Delta\sigma_i} + \mathcal{O}(\Delta\sigma_i^2) \text{ for } i = 1,2$$

for a small volatility perturbation $\Delta\sigma_i \ll 1$.

We note that the vega per bucket may possibly be negative for both the first and the second bucket. As an example of vega negativity, we compute the vega per bucket for the deal described in Exhibit A-1. Results are displayed in Exhibit A-2. The volatility is perturbed by a small amount.

The resulting vega is insensitive to either the perturbation size or the density of the 2D integration grid. In several instances a vega per bucket is negative, in both the first and the second bucket.

To ensure that the negative vega is not due to an implementation error, we develop an alternative valuation of the two-stock Bermudan option (available upon request). It is based on conditioning and involves a one-dimensional numerical integration over the Black formula. The alternative method yields the exact same results.

Note in Exhibit A-2 that the negative vegas occur in the case of high correlation and for the bucket with the lowest volatility. In the case of high correlation and one stock with

EXHIBIT A-1
Deal Description

Spot price for stock 1	$S_1(0)$	150
Spot price for stock 2	$S_2(0)$	140
Strike price for stock 1	K_1	100
Strike price for stock 2	K_2	100
Exercise time for stock 1	T_1	1Y
Exercise time for stock 2	T_2	2Y
Volatilities	σ_i	Variable
Correlation	ρ	0.9
Risk-free rate	r	5%

EXHIBIT A-2
Results for Negative Vega per Bucket for Two-Stock Bermudan Option

	σ_1	σ_2	price	$\mathcal{V}_1^{100bp}$	$\mathcal{V}_2^{100bp}$
Scenario 1	10%	30%	64.53	-0.45	0.56
Scenario 2	30%	10%	65.11	0.56	-0.44

significantly higher volatility than the other, we contend that the only added value of the additional option on the low-volatility stock lies in offering protection against a down move of both stocks (recall that the stocks are highly correlated). There are two scenarios:

- *Up move.* Both stocks move up. Because the high-volatility stock moves up much more than the low-volatility stock, the high-volatility call will be exercised.
- *Down move.* Both stocks move down. Because the high-volatility stock moves down much more than the low-volatility stock, the high-volatility call becomes out of the money, and the low-volatility call will be exercised.

If now the volatility of the low-volatility stock is increased by a small amount, then in these scenarios the exercise strategy remains unchanged. Also, in the case of an up move, the payoff remains unaltered. In the case of a down move, however, the low-volatility stock (volatility slightly increased) moves down more than in the unperturbed case. Therefore, the payoff of the protection call is reduced. In total, the Bermudan option is thus worth less.

ENDNOTES

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[1]A coterminal Bermudan swaption is an option to enter into an underlying swap at several exercise opportunities. The holder of a Bermudan swaption has the right at each exercise opportunity to either enter into a swap or hold the option; all the underlying swaps that may possibly be entered into have the same ending date.

[2]It was verified that the resulting vega is stable for a wide range of volatility perturbation. For very extreme perturbation, the vega is unstable. At high levels of perturbation, vega-gamma terms affect the vega. At too low levels of volatility perturbation, floating point number round-off errors affect the vega.

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