Generic Market Models[1]

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Abstract. Currently, there are two market models for valuation and risk management of interest rate derivatives, the LIBOR and swap market models. We introduce arbitrage-free constant maturity swap (CMS) market models and generic market models featuring forward rates that span periods other than the classical LIBOR and swap periods. We develop generic expressions for the drift terms occurring in the stochastic differential equation driving the forward rates under a single pricing measure. The generic market model is particularly apt for pricing of Bermudan CMS swaptions, fixed-maturity Bermudan swaptions, and callable hybrid coupon swaps.

Key words: generic market model, drift terms, BGM model, CMS

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1 Introduction

Currently, there are two types of market models for valuation and risk management of interest rate derivatives, which are the LIBOR and swap market models

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of Brace, Gątarek & Musiela (1997), Jamshidian (1997), Musiela & Rutkowski (1997) and Miltersen, Sandmann & Sondermann (1997). We introduce generic market models featuring forward rates that span periods other than the classical LIBOR and swap periods. The generic market model generalizes the LIBOR and swap market models. We derive necessary and sufficient conditions for the structure of the forward rates to span an arbitrage-free economy in terms of relative discount bond prices, at all times. We develop generic expressions for the drift terms occurring in the stochastic differential equation (SDE) driving the forward rates under a single pricing measure. We show how the instantaneous correlation of the generic forward rates can be calculated from the single instantaneous correlation matrix of forward LIBOR rates. These results are sufficient for implementation of calibration and pricing algorithms for generic market models.

Generic market models are specifically designed for pricing certain types of swaps. In particular, we consider constant maturity swaps (CMS) and hybrid coupon swaps. An interest rate swap is an agreement to exchange interest rate payments, for a specified period, frequency, and underlying notional. In a plain-vanilla swap, the floating interest rate is the LIBOR rate. In a *constant maturity swap* instead a swap rate is paid with specified tenor, fixed for all payments in the CMS swap. The payment frequency remains unchanged however. A *hybrid coupon swap* is a swap that features a floating payment schedule, designating the nature of the floating payments. A floating payment can be determined by either a LIBOR rate with varying maturity or a swap rate with varying tenor. An example of such a payment schedule is given in Table 1. Additionally, the function that transforms the LIBOR or swap rate into a cash flow may even be not entirely linear, for example, capped, floored or inverse.

The above swaps may have the feature that the swaps can be cancelled. Such versions are deemed *cancellable swaps.* To hold a cancellable swap is equal to holding a swap and an option to enter into the very same swap but with reversed cash flows[2]. The latter option is called a *callable swap.* In this paper we are also concerned with pricing callable and cancellable CMS and hybrid coupon swaps. There are two types of callable swaptions: *fixed-maturity* or *co-terminal.* A co-terminal option allows to enter into an underlying swap at several exercise opportunities, where each swap ends at the same contractually determined end date. The swap maturity becomes shorter as exercise is delayed. In contrast, for the fixed-maturity version, each underlying swap has the same contractually specified maturity and the respective end dates then differ.

The main outset of the paper is that a model is deemed proper for valuing a certain callable or cancellable swap, if the volatility of a rate that appears in the contract payoff has been calibrated correctly to the market volatility. The concept is best illustrated by example. In the case of the hybrid coupon swap of Table 1 at the valuation date 11 June 2004, we would want to calibrate exactly to the volatilities of the $1Y \times 2Y$ swaption, $2Y \times 4Y$ swaption, $3Y$ caplet, $4Y \times 2Y$ swaption and $5Y$ caplet. In contrast, for a cap one would calibrate to the

[2] Some readers might not be familiar with 'callable' and 'cancellable' swaps and might prefer to think of swaps and options thereon.

Table 1: Example of a hybrid coupon swap payment structure for the floating side. Date roll is modified following and day count is actual over 365.

Fixing date	Year fraction	Payment date	Rate
11-Jun-04	1.005479	13-Jun-05	1Y LIBOR
13-Jun-05	0.997260	12-Jun-06	2Y swap rate
12-Jun-06	0.997260	11-Jun-07	4Y swap rate
11-Jun-07	1.002740	11-Jun-08	1Y LIBOR
11-Jun-08	1.000000	11-Jun-09	2Y swap rate
11-Jun-09	1.000000	11-Jun-10	1Y LIBOR

volatilities of the $1Y$, $2Y$, $3Y$, $4Y$ and $5Y$ caplets. For a co-terminal Bermudan swaption, to the volatilities of the $1Y \times 5Y$, $2Y \times 4Y$, $3Y \times 3Y$, $4Y \times 2Y$ and $5Y \times 1Y$ swaptions. When employing a LIBOR market model to value a cap, the model would feature the following $1Y$ forward LIBOR rates: $1Y$, $2Y$, $3Y$, $4Y$ and $5Y$. If a swap market model would be used to value the Bermudan swaption, it would feature the $1Y \times 5Y$, $2Y \times 4Y$, $3Y \times 3Y$, $4Y \times 2Y$ and $5Y \times 1Y$ forward swap rates. For both LIBOR and swap market models, the canonical interest rates are simply equipped with the corresponding canonical volatilities, allowing for an efficient and straightforward calibration. Obviously, to straightforwardly calibrate a market model for the hybrid coupon swap of Table 1, and callable or cancellable versions thereof, the model would have to feature the forward swap rates $1Y \times 2Y$, $2Y \times 4Y$, $4Y \times 2Y$, and the $1Y$ forward LIBOR rates at $3Y$ and $5Y$. Up to now, whether a model containing such rates would be arbitrage-free is not well-known. To our knowledge, generic methods for deriving arbitrage-free drift terms for the SDE driving the various forward rates have not been developed yet. We develop such generic theory.

In terms of practical relevance, the generic market model technology is valuable to financial institutions that aim to trade in CMS Bermudan swaptions or callable hybrid coupon swaps. As such, their costumers might require any sequence of various maturity LIBOR or swap rate payments in the tailored exotic derivatives that they demand for their business. We show that a generic implementation of the resulting drift terms is feasible in practice, thereby enabling proper pricing and hedging of such hybrid coupon swaps.

A further motivation for the theory in this paper is that the idea of generic market models is not new to the finance literature, since it has already been suggested by Galluccio, Huang, Ly & Scaillet (2004). These authors discuss what they call the *co-sliding* (commonly referred to as 'LIBOR') and *co-terminal* (commonly referred to as 'swap') market models. The class of co-sliding market models corresponds to our class of CMS market models, but ours is defined differently. Galluccio et al. (2004) show that the only admissible co-sliding model is the LIBOR market model. Interestingly, we show that there are n arbitrage-free CMS market models associated with a tenor structure with n fixings, and the LIBOR and swap models are two special cases of these CMS models. In addition to the n CMS models, we introduce generic market models, extending the number of arbitrage-free market models to $n!$. Also, Galluccio et al. (2004) discuss the *co-initial* market model of Hunt & Kennedy (2000, Section 18.4), but this model does not fit into our dynamic market model framework. Moreover, in contrast to Galluccio et al. (2004), we derive generic expressions for the drift terms of the forward rates, for all $n!$ models (thus for LIBOR, swap, CMS and generic models).

An alternative way of calibrating a model to the relevant volatility levels, is to consider a LIBOR market model, and derive generic approximate expressions for the volatility of various forward rates. Such a procedure, for the specific case of calibration of the LIBOR model to swaption volatility, has been investigated in Jäckel & Rebonato (2003), Joshi & Theis (2002), Hull & White (2000) and Pietersz & Pelsser (2004). The advantage of the generic market model specification is that the relevant volatility functions can be directly specified. Moreover, the development of the theory of generic market models is justified already by the additional insight into the workings of LIBOR and swap market models.

We mention three areas of market model theory to which the generic market model approach extends. First, generic models may also be used in multi-currency market models, see Schlögl (2002). Second, a numerical implementation of a generic model may utilize drift approximations, see, for example, Hunter, Jäckel & Joshi (2001) and Pietersz, Pelsser & van Regenmortel (2004). Third, generic models may be equipped with smile dynamics. The volatility smile is the phenomenon that for European options different implied volatilities are quoted in the market when the strike of the option is varied. Generic market models do not restrict the instantaneous volatility in any way. As a result, smile-incorporating models can be readily applied to the generic market model framework. An application of smile modelling to the LIBOR market model can be found in Andersen & Andreasen (2000).

An outline of the paper is as follows. First, preliminaries are introduced. Second, necessary and sufficient no-arbitrage conditions on the structure and values of the forward rates are derived. Third, generic arbitrage-free drift terms are derived under a common measure. Fourth, the efficiency of drift calculations is discussed. Fifth, calibration of generic market models to correlation is addressed. Sixth, we end with conclusions.

2 Preliminaries

Consider a *tenor structure* $0 =: t_1 < \cdots < t_{n+1}$ and *day count fractions* α_i, over the period $[t_i, t_{i+1}]$, for $i = 1, \ldots, n$. Suppose traded in the market is a set of m forward LIBOR or swap rate agreements that are associated with that tenor structure[3]. Initially, m may be different from n, but in Theorem 1 we show that it makes sense, from an economic point of view, to consider only $m = n$. The

[3]The frequency of floating payments is restricted to one payment per fixed-payment period, but this is only for ease of exposition. In practice, this assumption may be relaxed; the theory is unchanged for any positive whole number of floating payments per fixed-payment period.

set of associated forward swap agreements is administered by a set of pairs

$$\mathcal{E} \;=\; \Big\{\, \epsilon_j = \big(s(j), e(j)\big)\ ;\ j = 1, \ldots, m\ ;\ s(j), e(j) \text{ integers}\ ; \\ 1 \le s(j) < e(j) \le n+1 \,\Big\}. \qquad (1)$$

Here $s(j)$ and $e(j)$ denote *start* and *end* of the forward swap agreement. The above set expression for $\mathcal{E}$ simply designates that there are m associated forward swap agreements, that each forward swap agreement starts and ends on one of the tenor times and that a start is strictly before an end. If the start s and end e of two forward swap agreements $\epsilon^{(1)}$, $\epsilon^{(2)}$ are equal, then $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are considered equal, thereby a priori excluding the possibility of different forward rates for the same forward swap agreement. Note also that different payment frequencies for a given swap period are not allowed. The value of the forward rate associated with ϵ_j is denoted by f_j. Forward rate f_j may, and shall, in the course of our paper, depend on time, $f_j = f_j(t)$. The associated forward swap agreement is defined as follows. At times $t_{s(j)}$ and $t_{e(j)}$ the agreement starts and ends, respectively. The agreement is partitioned by the $e(j)-s(j)$ accrual periods $[t_{s(j)}, t_{s(j)+1}]$, ..., $[t_{e(j)-1}, t_{e(j)}]$. The LIBOR rate is recorded at the start of each accrual period. If the accrual periods are indexed by $i = s(j), \ldots, e(j)-1$, then the LIBOR-observation time is t_i, the maturity of the LIBOR deposit is $t_{i+1}-t_i$, and the observed LIBOR rate is denoted by $\ell(t_i)$. If forward swap agreement j has been entered into at time t^* at rate $f_j(t^*)$, then the fixed and floating payments are $\alpha_i f_j(t^*)$ and $\alpha_i \ell(t_i)$, respectively. We assume liquid trading in the market at times $t^* = t_1, \ldots, t_n$ of those forward swap agreements $\epsilon \in \mathcal{E}$ for which $t_{s(j)} \ge t^*$. In other words, there is trading in a forward swap agreement if the agreement has not yet started or is about to start. We assume the cost of entering into any forward swap agreement at any tenor time to be zero.

The forward swap agreement structures of the LIBOR and swap market models fit into the framework of (1). For the LIBOR market model (LMM), $\mathcal{E}_{\mathrm{LMM}} = \{(1,2), (2,3), \ldots, (n, n+1)\}$. For the swap market model (SMM), $\mathcal{E}_{\mathrm{SMM}} = \{(1, n+1), (2, n+1), \ldots, (n, n+1)\}$. We introduce here a third kind of market model, associated with the q-period CMS rates. We name it the *CMS(q) market model*, for $q = 1, \ldots, n$, and it is defined by $\mathcal{E}_{\mathrm{CMS}(q)} = \{(1, 1+q), (2, 2+q), \ldots, (n-q+1, n+1), (n-q+2, n+1), \ldots, (n, n+1)\}$. Note that for $q = 1$ and $q = n$ we retain the LIBOR and swap market models, respectively.

The structure of these market models can be specified equivalently as follows, too. There exists an enumeration $\epsilon_j = (s(j), e(j))$, such that, for the LIBOR model, $s(j) = j$, $e(j) = j+1$. For the swap model, $s(j) = j$, $e(j) = n+1$. For the CMS(q) model, $s(j) = j$,

$$e(j) = j + q \;\; (j = 1, \ldots, n-q+1), \;\; e(j) = n+1 \;\; (j = n-q+2, \ldots, n). \qquad (2)$$

2.1 Absence of arbitrage

Associated with the tenor structure we also consider *discount bonds*. A discount bond is a hypothetical security that pays one unit of currency at its maturity.

The price at time t of a discount bond maturing at time t_i is denoted by $b_i(t)$. Note that there are $n+1$ discount bonds and that we necessarily have $b_i(t_i) = 1$ for $i = 1, \ldots, n+1$. The latter is just saying that the cost of immediately receiving one unit of currency is one unit of currency. The time-t_1 discount bond prices are sometimes simply denoted by b_i rather than by $b_i(t_1)$.

In terms of price consistency among discount bonds, forward swap agreements, and LIBOR deposits, we require some form of absence of arbitrage. We follow Musiela & Rutkowski (1997), in which two forms of no-arbitrage are introduced. First, a weaker notion of no-arbitrage is the usual no-arbitrage condition in a *pure bond market*. Second, a stronger notion of no-arbitrage assumes, in addition, that *cash* is also available in the market, which means that money, not stored in a money market account, can be carried over at zero cost. The stronger form of no-arbitrage excludes situations allowed by the weaker form. For example, discount bond prices greater than 1 (negative interest rates) are excluded by the strong form, but not by the weak form. More generally, discount bond prices are required, by the strong form, but not by the weak form, to not increase with increasing maturity, as shown by Musiela & Rutkowski (1997, page 267, below Equation (13)). In Section 3 below, it is shown that generic market models guarantee the weak form of no-arbitrage. The weak form is the natural condition for generic market models. Log-normal LIBOR models are known to satisfy the strong form of no-arbitrage; but the LIBOR model is a special case in this regard. For market models other than LIBOR, whether the strong form is satisfied is less clear. In fact, a multi-factor log-normal swap market model in general violates the strong form of arbitrage with positive probability, see Section 2.2 below. Therefore, hereafter we only consider the weak form of no-arbitrage, and any mentioning of 'no-arbitrage' refers to the weak form.

Definition 1 (Weak form no-arbitrage; static bond market) *Let* $\mathbf{x} = (x_1, \ldots, x_{n+1})$ *denote the holdings in discount bonds that mature respectively at times* $t_1 < \cdots < t_{n+1}$*. Prices of discount bonds at time* t_0 $(t_0 < t_1)$ *are denoted by* $\mathbf{b} = (b_1, \ldots, b_{n+1})$*. An arbitrage is a portfolio* $\mathbf{x}$ *such that:*

1. *The time-t_0 value is less than or equal to zero;* $\mathbf{b} \cdot \mathbf{x} \leq 0$*.*
2. *The time-t_i payoff of discount bond i is greater than or equal to zero;* $x_i \geq 0 \ \forall i$*.*
3. *There is at least one discount bond i that has a payoff at time t_i that is strictly greater than zero;* $\exists i : x_i > 0$*.*

A static bond market satisfies the weak form of no-arbitrage if no weak form arbitrage opportunities exist.

The following characterizes the weak form of no-arbitrage in the static case.

Lemma 1 *The weak form of no-arbitrage holds in a static bond market if and only if all discount bond prices are strictly positive;* $\mathbf{b} > 0$*.*

Proof: First, suppose $\mathbf{b} > 0$ and suppose there exists an arbitrage $\mathbf{x}$. From Definition 1, property 3, $\exists i : x_i > 0$. We have, for the time-t_0 value of $\mathbf{x}$:

$$\mathbf{b} \cdot \mathbf{x} = \underbrace{b_1 x_1 + \cdots + b_{i-1} x_{i-1} + b_{i+1} x_{i+1} + \cdots + b_{n+1} x_{n+1}}_{\geq 0} + \underbrace{b_i x_i}_{>0} > 0,$$

which is in contradiction to property 1 of the arbitrage portfolio $\mathbf{x}$.

Second, suppose there exists i such that $b_i \leq 0$. Consider the following portfolio $\mathbf{x}$, with $x_i = 1$ and $x_j = 0$ for $j \neq i$. Then the time-t_0 value of $\mathbf{x}$ is $b_i \leq 0$. Moreover, $\mathbf{x} \geq 0$ and $x_i > 0$, thus $\mathbf{x}$ is an arbitrage opportunity. □

The no-arbitrage referred to in this section and the next (Sections 2.1–3) relates to a static point in time; but this static no-arbitrage must of course hold at any point in time. Next to static no-arbitrage, there is dynamic no-arbitrage that relates to drift terms under a common probability measure; these no-arbitrage drifts are derived in Section 4 below.

2.2 Swap market models violate strong form no-arbitrage

We provide an example of a swap market model that violates the strong form of no-arbitrage with positive probability. We consider two co-terminal forward swap rates $f_{1:3}$, $f_{2:3}$, with tenor structure $t_0 < t_1 < t_2 < t_3$ and day count fractions α_1, α_2. At time t_1, we have the following equations, equating the values of fixed and floating legs:

$$\begin{aligned} \alpha_1 f_{1:3}(t_1) b_2(t_1) + \alpha_2 f_{1:3}(t_1) b_3(t_1) &= 1 - b_3(t_1), \\ \alpha_2 f_{2:3}(t_1) b_3(t_1) &= b_2(t_1) - b_3(t_1). \end{aligned}$$

Solving for the discount bond prices yields (suppressing dependency of t_1):

$$b_1 = \big(1 + \alpha_2 f_{2:3}\big) b_2, \quad b_2 = \frac{1}{1 + (\alpha_1 + \alpha_2) f_{1:3} + \alpha_1 \alpha_2 f_{1:3} f_{2:3}}.$$

If we assume $f_{1:3} \geq 0$, $f_{2:3} \geq 0$, then we find that $b_2 \leq 1$ is always true, however $b_1 \leq 1$ holds if and only if:

$$f_{2:3} \leq \frac{(\alpha_1 + \alpha_2) f_{1:3}}{\alpha_2 (1 - \alpha_1 f_{1:3})}, \quad (\alpha_1 f_{1:3} \leq 1). \tag{3}$$

If $b_1 > 1$, then the strong form of arbitrage is clearly violated. The region in which this occurs for $\alpha_1 = \alpha_2 = 1$ is displayed in Figure 1. When $f_{1:3}$, $f_{2:3}$ have log-normal dynamics with correlation $\rho < 1$, then, from (3), we find that there is always a positive probability that the forward rates $f_{1:3}$, $f_{2:3}$ end up in the region where the strong form of no-arbitrage is violated.

2.3 Dynamic market models

Valuation of non-European interest rate derivatives requires a dynamic model, i.e., a model that generates unique arbitrage-free discount bond prices at all future time points. A formal definition of dynamic market models is stated in Theorem 1 below. Examples of dynamic models are the LIBOR and swap market models. An example of a non-dynamic model is the co-initial market model, as defined by Hunt & Kennedy (2000, Section 18.4). The co-initial model features forward swap rates that span the periods $(1, 2)$,$(1, 3)$,…,$(1, n+1)$, that

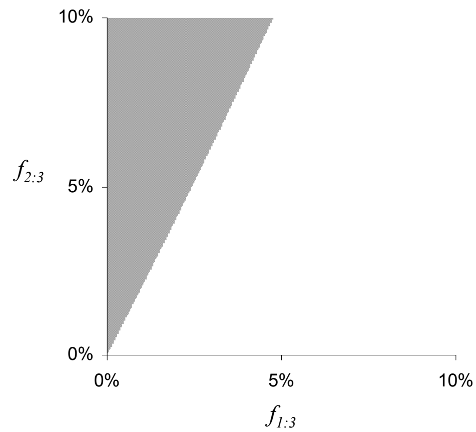

Figure 1: In the gray region the strong form of no-arbitrage is violated, $\alpha_1 = \alpha_2 = 1$.

is, all swap rates start at time t_1 but end consecutively at times $t_2, \ldots, t_{n+1}$. The co-initial model is non-dynamic since at time t_2, all forward swap agreements have expired. Though non-dynamic models are important, we restrict to examining dynamic market models only. We do so for restraining the length of the exposition. Also, the requirement that a market model be dynamic yields a compact characterization of such models in the form of Theorem 1 below.

For the dynamic case, arbitrary specification of forward rates at not only t_1, but at all time points $t_1, \ldots, t_n$, is required to lead to unique discount bond prices. Given an arbitrary set $\mathcal{E}$ of forward rates and their values $\{f_j(t_i)\}_{i,j}$, there are two mutually exclusive possibilities, given in the following definition.

Definition 2

- Condition A. *At each of the times $t_1, \ldots, t_n$, there is a unique system of prices for the discount bonds, such that the resulting aggregate trade system of discount bonds, forward swap agreements, and LIBOR deposits, is arbitrage-free.*
- Condition B. *At least at one of the times $t_1, \ldots, t_n$, either there exists no system or there are more than one different systems of prices for the discount bonds, such that the resulting aggregate trade system of discount bonds, forward swap agreements, and LIBOR deposits, is arbitrage-free.*

Obviously, we would want condition A to hold in financial models, and, in particular, in generic market models. We derive necessary and sufficient conditions on $\mathcal{E}$ and the values $\{f_j(t_i)\}$, for condition A to hold. In particular, given $n+1$ tenor times, we show that there are exactly $n!$ possibilities of choosing $\mathcal{E}$. The CMS market model (with LIBOR and swap models as special cases) only accounts for n of these possibilities. An example for $n = 6$ with market models of LIBOR, CMS(3), swap, co-initial, and the hybrid swap of Table 1 (viewed from the valuation date 11 June 2003), is given in Figures 2 and 3.

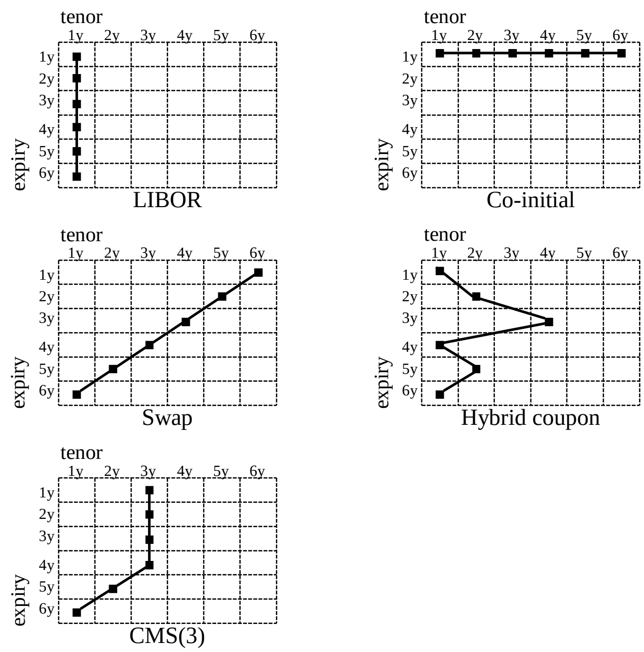

Figure 2: The swaptions from the swaption matrix to which various market models are calibrated.

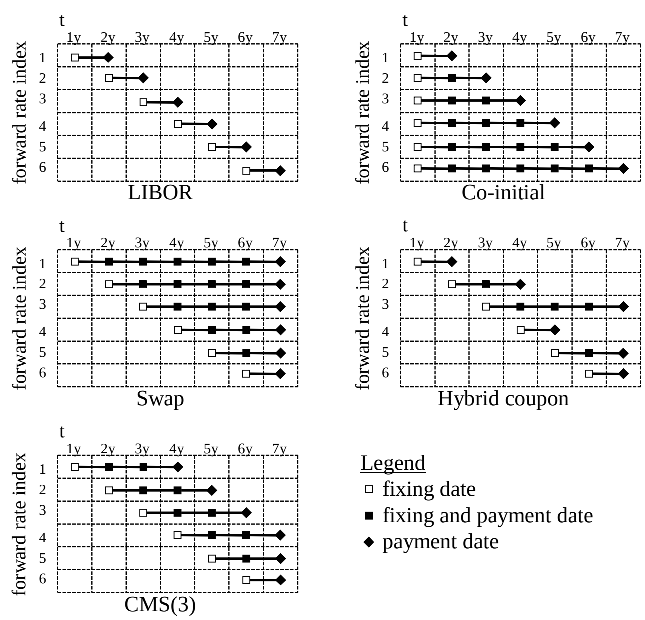

Figure 3: An overview of the forward swap agreements for various market models.

3 Necessary and sufficient conditions on the forward swap agreements structure for guaranteed no-arbitrage

We derive necessary and sufficient conditions for a set of forward rates to specify unique arbitrage-free discount bond prices. The program to achieve that goal is as follows. First, we value the forward swap agreements in terms of discount bond prices. Second, the conditions on the forward swap agreements are translated into conditions on the discount bond prices.

A forward swap agreement is valued by valuation of its floating and fixed payments in turn. The collections of floating and fixed payments of a forward swap agreement are called *floating* and *fixed legs*, respectively. The value $\pi_{\text{flt}}(\epsilon)$ of the floating leg of a forward swap agreement $\epsilon = (s, e)$ is[4]

$$\pi_{\text{flt}}(\epsilon) = b_s - b_e.$$

This equation can be seen to hold by considering a portfolio in the discount bonds that has the exact same cash flows as the floating leg, to wit, long a discount bond maturing at time t_s and short a bond maturing at time t_e. At time t_s, we invest the proceeds of the long position in the discount bond into the LIBOR deposit. At each LIBOR payment, we re-invest the notional into the LIBOR deposit. At the end of the floating leg, the notional cancels against the short position in the discount bond. It is not hard to see that such procedure provides the exact same cash flows as a floating leg.

The value $\pi_{\text{fxd}}(\epsilon, f)$ of a fixed leg with forward rate f can be obtained by simply discounting back the known future cash flows[5],

$$\pi_{\text{fxd}}(\epsilon, f) = f \underbrace{\sum_{i=s}^{e-1} \alpha_i b_{i+1}}\,.$$

The under-braced expression is also called *present value of a basis point* (*PVBP* in short), and is denoted by $p_{s:e}$.

The conditions on the forward rates are governed by the forward swap agreements to have zero value, that is, $\pi_{\text{flt}}(\epsilon) - \pi_{\text{fxd}}(\epsilon, f) = 0$. In fact, there exists a unique system of prices for the discount bonds consistent with the forward rates if and only if the system of m linear equations in the n unknown variables $b_2, \dots, b_{n+1}$ given by

$$\left\{ b_{s(j)} - b_{e(j)} - \sum_{i=s(j)}^{e(j)-1} f_j \alpha_i b_{i+1} = 0 \right\}_{j=1}^{m}, \tag{4}$$

[4] Here we assume equality of the forecast and discount curves and of the payment and index day count fractions.

[5] Note that we assume, for notational simplicity only, that the fixed payment frequency equals the floating payment frequency.

with $b_1 = 1$, has a unique solution. The latter is already a precisely specified and tractable necessary and sufficient condition for existence of unique discount bond prices that are consistent with the forward rates. This condition can be validated by numerically checking invertibility of linear equation (4). In the sequel, we develop conditions and implications that are more straightforward to verify and that a priori guarantee invertibility of (4), and we sketch scenarios in which these implications hold. It is shown that invertibility of (4) is guaranteed in typical finance scenarios, and that invertibility can be violated only under extreme situations, that are fully irrelevant to a finance setting.

The following assumption on the values that forward rates can attain allows us to establish the weak form of no-arbitrage for generic market models.

Assumption 1 *A forward rate f can only attain non-negative values: $f \geq 0$.*

Assumption 1 is satisfied almost always in any interest rate market. Only in very rare occasions have negative interest rates been observed. An example of negative interest rates in Japan at the start of November 1998 is given in Ostrom (1998). These interest rates reached -3 to -6 basis points (bp) (-.03% to -.06%). Moreover, the popular displaced diffusion smile model of Rubinstein (1983) can generate negative forward rates with positive probability, if the displacement parameter is negative. However, violation of Assumption 1 does not necessarily imply that the system of forward rates admits arbitrage of the weak form. In fact, we make plausible that slightly negative interest rates still allow for unique discount bond prices that are arbitrage-free in the weak sense, by considering a simple numerical example. Consider a single forward rate, two tenor times $\{t_1 = 0, t_2\}$ market model. The price of the discount bond for maturity at time t_2 is given by $1/(1 + \alpha f)$. The rate f should thus satisfy $f > -1/\alpha$, to ensure a positive and finite price for the discount bond. For annual payments, for which $\alpha \approx 1$, we have $-1/\alpha \approx -100\%$. In fact, for more frequent payments than annual, the arbitrage-defying rate is even more negative than -100%. These considerations lead us to conclude that arbitrage of the weak form in a forward swap agreement market can occur only in situations that are considered financially extreme. Essential to no-arbitrage is thus the structure of the forward swap agreements.

3.1 Main result

The main result can now be formulated. The theorem below states that, for dynamic market models, (i) if a tenor structure has n fixing times $t_1, \ldots, t_n$, then we require n forward swap agreements, and (ii) for each fixing time t_i, there is exactly one forward swap agreement that starts at that fixing time t_i, $i = 1, \ldots, n$. Note that the co-initial model does not fit the requirements below, though it is a perfectly sensible arbitrage-free model. The reason that the co-initial model is not incorporated is the requirement that a model be *dynamic*, see the discussion in Section 2.1.

Theorem 1 *Let $\{t_1, \ldots, t_{n+1}\}$ be a set of tenor times. Let $\mathcal{E} = \{\epsilon_j\}_{j=1}^{m}$ and f_j be a set of forward swap agreements and forward rates, respectively, associated*

Algorithm 1 Back substitution.

Input: n, $\mathbf{U}$ ($(n+1)\times(n+1)$ unit upper-triangular), $\mathbf{c}\in\mathbb{R}^{n+1}$.
Output: $\hat{\mathbf{b}}=\mathbf{U}^{-1}\mathbf{c}\in\mathbb{R}^{n+1}$.

1: Set $\hat{b}_{n+1}\Leftarrow c_{n+1}$.
2: **for** $i=n,\ldots,1$ **do**
3: $\quad\hat{b}_i\Leftarrow c_i-\sum_{j=i+1}^{n+1}u_{ij}\hat{b}_j$.
4: **end for**

with the tenor times. Then, at each of the times $t_1,\ldots,t_n$, for all forward rates $\{f_j\}_{j=1}^m$ satisfying Assumption 1, there exists a unique weak-form arbitrage-free solution to the system of linear equations (4) in the discount bond prices, if and only if $m=n$ and there exists an ordering of the n forward swap agreements $\epsilon_j=(s(j),e(j))$, $j=1,\ldots,m$ such that $s(j)=j$.

Proof: The proof is split into two parts. First, we prove that the described structure leads to arbitrage-free invertibility of system (4) for all forward rates satisfying Assumption 1. Second, the reverse implication is proven.

Suppose that the structure $\mathcal{E}$ of forward swap agreements is such that $m=n$ and that an ordering of the n forward swap agreements $\epsilon_j=(s(j),e(j))$, $j=1,\ldots,m$ exists such that $s(j)=j$. The existence of unique arbitrage-free discount bond prices is guaranteed if we show unique discount bond prices exist that are all positive, see Lemma 1. To that order, consider system (4) in terms of the deflated discount bond prices, $\hat{b}_i\equiv b_i/b_{n+1}$, and substitute $s(j)=j$,

$$\left\{\ \hat{b}_j-\hat{b}_{e(j)}-\sum_{i=j}^{e(j)-1}f_j\alpha_i\hat{b}_{i+1}=0\ \right\}_{j=1}^{n},\ \{\hat{b}_{n+1}=1\}. \tag{5}$$

Note that the $(n+1)\times(n+1)$ matrix $\mathbf{U}=\mathbf{U}(\mathbf{f})$ associated with this system is unit upper-triangular, which means that the diagonal contains ones and that the lower-triangular part of the matrix contains zeros. It follows that this matrix is invertible. We thus have

$$\mathbf{U}(\mathbf{f})\hat{\mathbf{b}}=\mathbf{c},\ \ \hat{\mathbf{b}}=\mathbf{U}(\mathbf{f})^{-1}\mathbf{c},\ \ \mathbf{c}=(0\ \cdots\ 0\ 1)^T\in\mathbb{R}^{n+1}.$$

An efficient method for calculating the inverse of a unit upper-triangular matrix is *back substitution*, see for example Golub & van Loan (1996, Algorithm 3.1.2). Back substitution aids in the proof, therefore it is displayed in Algorithm 1. We show by induction for $i=n+1,n,\ldots,1$ that $\hat{b}_i\geq 1$. For $i=n+1$, $\hat{b}_i=\hat{b}_{n+1}=1$, by line 1 of Algorithm 1, which states that $\hat{b}_{n+1}=c_{n+1}=1$. Suppose, then, that $\hat{b}_j\geq 1$ for $j=i+1,\ldots,n+1$. We have, by line 3 of Algorithm 1, that $\hat{b}_i=c_i-\sum_{j=i+1}^{n+1}u_{ij}\hat{b}_j=-\sum_{j=i+1}^{n+1}u_{ij}\hat{b}_j$. Note that, for $j>i$, u_{ij} is either $-\alpha_j f_i$, $-1-\alpha_j f_i$, or 0. It follows that

$$\hat{b}_i=\underbrace{f_i\sum_{j=i}^{e(i)-1}\alpha_j\hat{b}_{j+1}}_{\geq 0}+\underbrace{\hat{b}_{e(i)}}_{\geq 1}\geq 1,$$

which concludes the induction proof. The unique solution for the *un*deflated discount bond prices at tenor point t_1 is then given by $b_i \equiv \hat{b}_i/\hat{b}_1$, which is defined and positive since $\hat{\mathbf{b}} = (\hat{b}_1, \dots, \hat{b}_{n+1}) \geq 1$.

Note that the above proof is independent of the number of tenor times. Therefore the forward swap agreements structure $n = m$ and $\{s(j) = j\}$ guarantees existence of unique arbitrage-free discount bond prices for *all* forward rates satisfying Assumption 1 at all tenor times $t_1, \dots, t_n$, which was to be shown.

The reverse implication is proven by induction on n. For $n = 1$, the result is immediate. Now, assume the result is true for $i = 1$ to $n-1$. We want to prove it is true for n. The model viewed from t_2 has n tenor points, so by the induction hypothesis we must have that: (i) $m \geq n-1$, (ii) there are exactly $n-1$ forward swap agreements that start at t_2 or later, (iii) for these $n-1$ forward swap agreements, there is an enumeration $j = 2, \dots, n$, such that $s(j) = j$. There are three possibilities: $m = n-1$, $m > n$ or $m = n$. We show that the cases $m = n-1$ and $m > n$ lead to non-uniqueness or non-invertibility of (4) for some of the forward rates f that satisfy Assumption 1.

If $m = n-1$, there are less equations than unknown variables in (4), and it follows that, if there is a solution at all, it will be non-unique.

If $m > n$, then we may form a sub-model with n forward swap agreements such that $s(j) = j$ for $j = 1, \dots, n$. We have already proven that such a structure with n forward rates leads to unique positive discount bond prices. For a left out forward swap agreement, say $\epsilon = (s, e)$, the associated forward rate f should then satisfy

$$f = \frac{b_s - b_e}{\sum_{i=s}^{e-1} \alpha_i b_{i+1}}. \tag{6}$$

We conclude then that there are forward rates satisfying Assumption 1 for which there do not exist discount bond prices.

Thus we must have $m = n$ and for remaining forward swap agreement 1 we have $s(1) = 1$ from which the result follows. □

As a corollary, we can count the dynamic market model structures given the number of tenor times $n+1$. For forward rate 1, we can chose from n end times $t_2, \dots, t_{n+1}$, for forward rate 2, from $n-1$ end times $t_3, \dots, t_{n+1}$, etcetera.

Corollary 1 (Counting dynamic market models) *Consider market models with $n+1$ tenor times. Then there are $n!$ ways of selecting forward swap agreements such that, for all forward rates satisfying Assumption 1, and at all tenor times $t_1, \dots, t_n$, there exist unique weak-form arbitrage-free discount bond prices satisfying (4).*

Note that Theorem 1 rules out the applicability of generic market models to Bermudan-callable spread options, in the sense that we cannot define two rates, fixing at the same time, as state variables.

4 Generic expressions for no-arbitrage drift terms

We derive generic expressions for the arbitrage-free drift terms of generic market models, that are so characteristic for the LIBOR and swap market models. We assume given a dynamic market model, therefore the forward swap agreements are of the form $\epsilon_i = (i, e(i))$. If dependency of the end index is clear we simply write $e(i)$ as e. The forward rate $f_{i:e}$ has start date t_i and end date t_e, and is modelled under its forward measure, associated with the PVBP $p_{i:e}$ as numeraire:

$$\frac{\mathrm{d}f_{i:e}(t)}{f_{i:e}(t)} = \boldsymbol{\sigma}_{i:e}(t) \cdot \mathrm{d}\mathbf{w}^{(i:e)}(t), \tag{7}$$

with $\boldsymbol{\sigma}_{i:e}$ a d-dimensional volatility vector, and with $\mathbf{w}^{(i:e)}$ a d-dimensional Brownian motion under the forward measure $\mathbb{Q}_{i:e}$ associated with $p_{i:e}$ as numeraire. The integer $d > 0$ is deemed the *number of factors* of the model. The volatility vector $\boldsymbol{\sigma}_{i:e}(t) = \boldsymbol{\sigma}_{i:e}(t, \omega)$ can be state dependent to allow for smile modelling.

For pricing of non-standard interest rate derivatives, it is necessary to derive dynamics for all forward rates under a common measure. We can work either with the terminal or spot measure. Each is treated below consecutively.

4.1 Terminal measure

We work with the terminal measure $\mathbb{Q}_{n+1}$, that is the measure associated with the terminal discount bond b_{n+1} as numeraire. Without loss of generality, the presentation is given as if all forward rates have not yet expired. We work with the numeraire-deflated discount bond prices. The quantity $\hat{p}_{i:e}$ denotes the deflated PVBP, $\hat{p}_{i:e} \equiv p_{i:e}/b_{n+1}$. The deflated PVBPs can be calculated, in turn, when the deflated discount bond prices $\hat{b}_i \equiv b_i/b_{n+1}$ are known. The deflated discount bond prices are given by (5). Recall that (5) can be written in matrix form as $\mathbf{U}\hat{\mathbf{b}} = \mathbf{c}$, with $\mathbf{c} = (0 \ \cdots \ 0 \ 1)^T$, and $\mathbf{U} = \mathbf{U}(\mathbf{f})$ an $(n+1) \times (n+1)$ unit upper-triangular matrix, given by

$$u_{ij} = \begin{cases} 0 & \text{if } i > j \text{ or } (i < j \text{ and } j > e(i)), \\ 1 & \text{if } i = j, \\ -\alpha_{j-1} f_{i:e(i)} & \text{if } i < j \text{ and } j < e(i), \\ -\alpha_{j-1} f_{i:e(i)} - 1 & \text{if } i < j \text{ and } j = e(i). \end{cases}$$

Thus $\hat{\mathbf{b}} = \mathbf{U}(\mathbf{f})^{-1}\mathbf{c}$. Write $\hat{\mathbf{p}}$ as a function of the forward rates, $\hat{\mathbf{p}} = \hat{\mathbf{p}}(\mathbf{f})$:

$$\hat{\mathbf{p}} = \mathbf{A}\hat{\mathbf{b}} = \mathbf{A}\mathbf{U}(\mathbf{f})^{-1}\mathbf{c}, \quad \mathbf{A} \equiv \begin{pmatrix} 0 & (\alpha_1 \ \cdots \ \alpha_{e(1)-1} \ 0 \ \cdots \ 0) \\ 0 \ \ 0 & (\alpha_2 \ \cdots \ \alpha_{e(2)-1} \ 0 \ \cdots \ 0) \\ 0 \ \ \vdots & \ddots \quad \ddots \qquad \vdots \\ 0 \ \ 0 & \cdots \quad 0 \qquad (\alpha_n) \end{pmatrix},$$

for the $n \times (n+1)$ matrix $\mathbf{A}$. Subsequently, define the Radon-Nikodým density

$$z_{i:e,n+1}(t) \equiv \frac{p_{i:e}(t)/b_{n+1}(t)}{p_{i:e}(0)/b_{n+1}(0)} = \frac{\hat{p}_{i:e}(t)}{\hat{p}_{i:e}(0)}. \tag{8}$$

Note that $z_{i:e,n+1}(t)$ is a martingale under the terminal measure $\mathbb{Q}_{n+1}$. This implies that

$$\frac{\mathrm{d}z_{i:e,n+1}(t)}{z_{i:e,n+1}(t)} = \frac{\mathrm{d}\hat{p}_{i:e}(t)}{\hat{p}_{i:e}(t)} = \boldsymbol{\theta}_{i:e,n+1}(t) \cdot \mathbf{w}^{(n+1)}(t), \tag{9}$$

with the d-dimensional vector $\boldsymbol{\theta}$ given by

$$\boldsymbol{\theta}_{i:e,n+1}(t) = \frac{1}{\hat{p}_{i:e}(t)} \sum_{k=i+1}^{n} \frac{\partial \hat{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) \boldsymbol{\sigma}_{k:e(k)}(t). \tag{10}$$

The summation is required only from $i+1$ to n since $\hat{p}_{i:e}$ is dependent on $f_{k:e(k)}$ only for $k > i$. Finally we apply Girsanov's theorem to obtain the required expression for $\mathrm{d}\mathbf{w}^{(i:e)}(t) - \mathrm{d}\mathbf{w}^{(n+1)}(t)$,

$$\mathrm{d}\mathbf{w}^{(i:e)}(t) - \mathrm{d}\mathbf{w}^{(n+1)}(t) = -\boldsymbol{\theta}_{i:e,n+1}(t)\mathrm{d}t. \tag{11}$$

Thus,

$$\begin{aligned} \frac{\mathrm{d}f_{i:e}(t)}{f_{i:e}(t)} &= -\frac{1}{\hat{p}_{i:e}(t)} \sum_{k=i+1}^{n} \frac{\partial \hat{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) |\boldsymbol{\sigma}_{k:e(k)}(t)||\boldsymbol{\sigma}_{i:e}(t)| \rho_{k:e(k),i:e}(t)\mathrm{d}t \\ &\quad + \boldsymbol{\sigma}_{i:e}(t) \cdot \mathrm{d}\mathbf{w}^{(n+1)}(t), \end{aligned} \tag{12}$$

where the scalar $\rho_{k:e(k),i:e}$ has been defined as

$$\rho_{k:e(k),i:e}(t) = \frac{\boldsymbol{\sigma}_{k:e(k)}(t) \cdot \boldsymbol{\sigma}_{i:e}(t)}{|\boldsymbol{\sigma}_{k:e(k)}(t)||\boldsymbol{\sigma}_{i:e}(t)|},$$

and has the interpretation of instantaneous correlation.

An expression is given for $\partial \hat{\mathbf{p}} / \partial f_{k:e(k)}$. Note that $\partial \mathbf{U} / \partial f_{k:e(k)}$ is a matrix that is zero bar a single row, the k^{th} row, and that the derivative is independent of f, since all f terms occur linearly in the matrix $\mathbf{U}$. The k^{th} row is filled, from entry $(k, k+1)$, with the row vector $(-\alpha_k \ \cdots \ -\alpha_{e(k)-1} \ 0 \ \cdots \ 0)$. We have that

$$\frac{\partial \hat{\mathbf{p}}}{\partial f_{k:e(k)}} = -\mathbf{A}\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \mathbf{U}^{-1}\mathbf{c} = -\mathbf{A}\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \hat{\mathbf{b}} = \mathbf{A}\mathbf{U}^{-1}\mathbf{c}_k \hat{p}_{k:e(k)}, \tag{13}$$

where $\mathbf{c}_k \in \mathbb{R}^{n+1}$ denotes the standard basis vector with unit k^{th} coordinate, and zero coordinates otherwise. We define $\tilde{b}_i^{(k)}$ by

$$\tilde{b}_i^{(k)} = (\mathbf{U}^{-1}\mathbf{c}_k)_i, \quad i = 1, \ldots, n, \quad k = 1, \ldots, n. \tag{14}$$

Substituting (14) into (13) yields

$$\frac{\partial \hat{p}_{i:e}}{\partial f_{k:e(k)}} = 1_{\{k \geq i+1\}} \hat{p}_{k:e(k)} \left(\sum_{j=i}^{\min(e(i)-1, k-1)} \alpha_j \tilde{b}_{j+1}^{(k)} \right). \tag{15}$$

Define $\mu(i,k) \equiv \min\big(e(i)-1, k-1\big)$. Substituting (15) into (12), suppressing the dependency of time, and using $\hat{p}_{k:e(k)} f_{k:e(k)} = \hat{b}_k - \hat{b}_{e(k)}$, we obtain the generic market model SDE under the terminal measure:

$$\frac{\mathrm{d}f_{i:e}}{f_{i:e}} = -\frac{1}{\hat{p}_{i:e}} \sum_{k=i+1}^{n} \big(\hat{b}_k - \hat{b}_{e(k)}\big) \left(\sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} \right) \boldsymbol{\sigma}_{k:e(k)} \cdot \boldsymbol{\sigma}_{i:e} \mathrm{d}t + \boldsymbol{\sigma}_{i:e} \cdot \mathrm{d}\mathbf{w}^{(n+1)}. \tag{16}$$

4.2 Spot measure

We work with the spot measure $\mathbb{Q}_{\text{Spot}}$, that is the measure associated with the spot LIBOR numeraire, defined as follows. The account starts out with one unit of currency. Subsequently, this amount is invested in the spot LIBOR account. After the first accrual period, the proceeds are re-invested in the then spot LIBOR account. This procedure is repeated. For the spot measure it is convenient to define the *spot index* $i(t)$, defined by $i(t) = \min\{\text{integer } \ i \ ; \ t < t_i\}$.

For the spot measure, we work with discount bond prices, deflated by the spot discount bond $b_{i(t)}$. The quantities $\bar{\mathbf{p}}$ and $\bar{\mathbf{b}}$ denote the vectors of $b_{i(t)}$-deflated PVBPs and discount bond prices, respectively. We have $\bar{\mathbf{p}} = \mathbf{A}\bar{\mathbf{b}}$ and

$$\bar{\mathbf{b}} = \frac{1}{\hat{b}_{i(t)}} \hat{\mathbf{b}} = \frac{1}{(\mathbf{U}^{-1}\mathbf{c})_{i(t)}} \mathbf{U}^{-1}\mathbf{c}.$$

The Radon-Nikodým density $z_{i:e,i(t)}(t)$ is defined similarly to (8). A martingale SDE for the Radon-Nikodým density holds,

$$\frac{\mathrm{d}z_{i:e,i(t)}(t)}{z_{i:e,i(t)}(t)} = \frac{\mathrm{d}\bar{p}_{i:e,i(t)}(t)}{\bar{p}_{i:e,i(t)}(t)} = \boldsymbol{\theta}_{i:e,i(t)}(t) \cdot \mathrm{d}\mathbf{w}^{(i(t))},$$

similar to (9), with d-dimensional volatility vector equal to

$$\boldsymbol{\theta}_{i:e,i(t)}(t) = \frac{1}{\bar{p}_{i:e}(t)} \sum_{k=i(t)}^{n} \frac{\partial \bar{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) \boldsymbol{\sigma}_{k:e(k)}(t). \tag{17}$$

Comparing (17) to (10), we find that, for the spot measure, we sum over all available forward rates from $i(t)$ to n, since $\bar{p}_{i:e}$ might depend on all those forward rates. Recall that, for the terminal measure, we need only sum from $i+1$ to n.

Similar to (11), we have $\mathrm{d}\mathbf{w}^{(i:e)} - \mathrm{d}\mathbf{w}^{(i(t))} = -\boldsymbol{\theta}_{i:e,i(t)}\mathrm{d}t$. Thus we obtain the equivalent of (12),

$$\begin{aligned} \frac{\mathrm{d}f_{i:e}(t)}{f_{i:e}(t)} &= -\frac{1}{\bar{p}_{i:e}(t)} \sum_{k=i(t)}^{n} \frac{\partial \bar{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) |\boldsymbol{\sigma}_{k:e(k)}(t)||\boldsymbol{\sigma}_{i:e}(t)| \rho_{k:e(k),i:e}(t)\mathrm{d}t \\ &\quad + \boldsymbol{\sigma}_{i:e}(t) \cdot \mathrm{d}\mathbf{w}^{(i(t))}(t). \end{aligned} \tag{18}$$

An expression for $\partial\bar{\mathbf{p}}/\partial f_{k:e(k)}$ is given by

$$\frac{\partial \bar{\mathbf{p}}}{\partial f_{k:e(k)}} = \frac{1}{\hat{b}_{i(t)}} \frac{\partial \hat{\mathbf{p}}}{\partial f_{k:e(k)}} + \frac{1}{\hat{b}_{i(t)}} \underbrace{\left(\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \mathbf{U}^{-1}\mathbf{c} \right)_{i(t)}}_{=\hat{p}_{k:e(k)} \tilde{b}_{i(t)}^{(k)}} \bar{\mathbf{p}}. \tag{19}$$

Similar as in (13) and (15) for the terminal measure, we find for the spot measure:

$$\frac{\partial \bar{p}_{i:e}}{\partial f_{k:e(k)}} = 1_{\{k \geq i+1\}} \bar{p}_{k:e(k)} \sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} - \bar{p}_{k:e(k)} \bar{p}_{i:e} \tilde{b}_{i(t)}^{(k)}. \tag{20}$$

Substituting (20) into (18), suppressing the dependency of time, and using $\bar{p}_{k:e(k)} f_{k:e(k)} = \bar{b}_k - \bar{b}_{e(k)}$, we obtain the generic market model SDE under the spot measure:

$$\frac{\mathrm{d}f_{i:e}}{f_{i:e}} = -\frac{1}{\bar{p}_{i:e}} \sum_{k=i(t)}^{n} \left(\bar{b}_k - \bar{b}_{e(k)}\right) \left(1_{\{k \geq i+1\}} \sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} - \bar{p}_{i:e} \tilde{b}_{i(t)}^{(k)}\right) \boldsymbol{\sigma}_{k:e(k)} \cdot \boldsymbol{\sigma}_{i:e} \mathrm{d}t + \boldsymbol{\sigma}_{i:e} \cdot \mathrm{d}\mathbf{w}^{(i(t))}. \tag{21}$$

4.3 An example: The LIBOR market model

For illustration, LIBOR drift terms are calculated starting from the generic market model framework. We stress here that the explicit calculations below of the generic expressions of the previous section are *not* required for implementation of a generic market model, but are merely performed for illustration only.

First, we derive the LIBOR SDE for the terminal measure, by applying (16). In the LIBOR market model, a forward rate $f_{k:e(k)}$ is denoted by f_k. Note that:

(i) $\hat{p}_{i:e(i)} = \hat{p}_{i:i+1} = \alpha_i \hat{b}_{i+1}$,

(ii) $\mu(i,k) = \min(e(i)-1, k-1) = \min(i, k-1) = i$, for $k = i+1, \ldots, n$,

(iii) $\tilde{b}_j^{(k)} = \frac{\hat{b}_j}{\hat{b}_k} 1_{\{j \leq k\}} = \frac{\bar{b}_j}{\bar{b}_k} 1_{\{j \leq k\}}$,

(iv) $\frac{\hat{b}_k - \hat{b}_{k+1}}{\hat{b}_k} = \frac{\bar{b}_k - \bar{b}_{k+1}}{\bar{b}_k} = 1 - \frac{1}{1+\alpha_k f_k} = \frac{\alpha_k f_k}{1+\alpha_k f_k}$,

(v) $\sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} = \frac{\hat{p}_{i:e(i)}}{\hat{b}_k} = \frac{\bar{p}_{i:e(i)}}{\bar{b}_k}$.

Substituting (i)–(v) into (16), we obtain,

$$\frac{\mathrm{d}f_i}{f_i} = -\sum_{k=i+1}^{n} \frac{\alpha_k f_k}{1+\alpha_k f_k} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_i \mathrm{d}t + \boldsymbol{\sigma}_i \cdot \mathrm{d}\mathbf{w}^{(n+1)},$$

which is the familiar SDE of the LIBOR model under the terminal measure.

Second, we derive the LIBOR SDE for the spot measure. If we substitute (i)–(v) into (21), we see that for $k \geq i+1$, $\sum_{j=i}^{i} \alpha_j \tilde{b}_{j+1}^{(k)}$ cancels against $\bar{p}_{i:i+1} \tilde{b}_{i(t)}^{(k)}$, and for $k \leq i$, we are left with $-\bar{p}_{i:i+1} \tilde{b}_{i(t)}^{(k)}$, therefore:

$$\frac{\mathrm{d}f_i}{f_i} = \sum_{k=i(t)}^{i} \frac{\alpha_k f_k}{1+\alpha_k f_k} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_i \mathrm{d}t + \boldsymbol{\sigma}_i \cdot \mathrm{d}\mathbf{w}^{(i(t))},$$

which is the familiar SDE of the LIBOR model under the spot measure.

5 Complexity of CMS market models

We study the complexity of drift calculations over a single time step. The LIBOR market model has a special structure that renders the complexity to $\mathcal{O}(nd)$, shown by Joshi (2003). We show that a similar approximate algorithm can be defined for CMS(q) market models, for the terminal measure. The algorithm is shown to be exact for the swap market model ($q = n$). The following quantity that occurs in the drift term is approximated:

$$\tilde{p}^{(k)}_{i:\mu(i,k)+1} = \tilde{p}^{(k)}_{i:\min(k,i+q)} := \sum_{j=i}^{\min(k,i+q)-1} \alpha_j \tilde{b}^{(k)}_{j+1} \quad (i < k). \tag{22}$$

The approximation is based on the assumption that α_i is close to α_{i+q}, for $i = 1, \ldots, n-q$. Note that this assumption is used only to efficiently approximate (22) for calculation of drift terms, and this assumption is *not* used in the calculation of contract payoffs. Moreover, if needs be, the drift terms can be calculated exactly by exact calculation of (22).

Approximation 1 *Approximately, by assumption of $\alpha_i \approx \alpha_{i+q}$ ($i = 1, \ldots, n-q$), we have, for $\tilde{p}^{(k)}_{i:\mu(i,k)+1}$ defined in (22),*

$$\tilde{p}^{(k)}_{i:\mu(i,k)+1} \approx \alpha_{k-1} \prod_{m=i}^{k-2} \left(1 + \alpha_m f_{m+1:e(m+1)} \right) \quad (i < k). \tag{23}$$

Here, an empty product denotes 1. Formula (23) is exact for $i > k - q - 1$. In particular, (23) is exact for any i in the swap market model ($q = n$).

The rationale for Approximation 1, as well as the proof of exactness when $i > k - q - 1$, are given in Appendix A. Note that accumulating errors in (23) are likely to cancel, since in practice the difference $\alpha_i - \alpha_{i+q}$ is both negative and positive. From (16) and Approximation 1, we obtain,

$$\frac{\mathrm{d}f_{i:e}}{f_{i:e}} \approx -\frac{1}{\hat{p}_{i:e}} \sum_{k=i+1}^{n} \big(\hat{b}_k - \hat{b}_{e(k)}\big)\alpha_{k-1} \prod_{m=i}^{k-2} \big(1 + \alpha_m f_{m+1:e(m+1)}\big) \boldsymbol{\sigma}_{k:e(k)} \cdot \boldsymbol{\sigma}_{i:e} \mathrm{d}t + \boldsymbol{\sigma}_{i:e} \cdot \mathrm{d}\mathbf{w}^{(n+1)}. \tag{24}$$

Define

$$\mathbf{v}_i = \sum_{k=i+1}^{n} \big(\hat{b}_k - \hat{b}_{e(k)}\big)\alpha_{k-1} \prod_{m=i}^{k-2} \big(1 + \alpha_m f_{m+1:e(m+1)}\big) \boldsymbol{\sigma}_{k:e(k)}. \tag{25}$$

The proof of the following lemma is deferred to Appendix B.

Lemma 2 *The quantity $\mathbf{v}_i$ defined in (25) satisfies the following recursive formulas:*

- $\mathbf{v}_n = \mathbf{0}$,

Table 2: Test description of exact versus approximate drifts in CMS(q) models.

Currency:	USD
Market data:	Swap rates and at-the-money swaption volatility
Valuation date:	18 July 2003
Deal:	30 year fixed-maturity Bermudan swaption
Start date:	16 June 2004
Index reference:	Annual, ACT/365, Modified following
Fixed coupon:	3.2%

- $\mathbf{v}_i = \big(1 + \alpha_i f_{i+1:e(i+1)}\big)\mathbf{v}_{i+1} + \alpha_i\big(\hat{b}_{i+1} - \hat{b}_{e(i+1)}\big)\boldsymbol{\sigma}_{i+1:e(i+1)}$.

In Algorithm 2 an $\mathcal{O}(nd)$ algorithm, based on Lemma 2, is displayed that approximately calculates the forward swap rates for a time step under the terminal measure, for the CMS(q) market model. This algorithm is exact for the swap market model ($q = n$). Algorithm 2 also calculates time-t values for discount bond prices (denoted by β) and for PVBPs ($p_{i:e(i)}$ is denoted by ϖ_i).

To benchmark the accuracy of Algorithm 2, various fixed-maturity Bermudan swaptions are priced in their corresponding CMS(q) market models, with both exact SDE (16) and approximate SDE (24). The deal specification is given in Table 2. The swap tenor is q years, with $31-q$ exercise opportunities, at (16 June 2004 + i years), $i = 0, \ldots, 30 - q$, for $q = 1, \ldots, 30$. The difference between the minimum (0.996) and maximum (1.007) attained day count fractions is 0.011. To price fixed-maturity Bermudan swaptions in Monte Carlo, we use the algorithm of Longstaff & Schwartz (2001), with the swap value as explanatory variable x, and basis functions 1, x and x^2. An 8 factor model is used ($d = 8$), with the correlation of the forward CMS(q) rates given by the parametrization of Rebonato (1998, Equation (4.5), page 83), $\exp(-\beta|t_i - t_j|)$, for rates $f_{i:e(i)}$ and $f_{j:e(j)}$, with $\beta = 3\%$. The differences between the prices obtained with exact and approximate drift terms are displayed in Figure 4. We note that for $q = n$, equal prices are obtained up to all digits. The results show that the error is small, up to only 3 bp of the option premium, and up to only 6% of the simulation standard error. Moreover, the error fluctuates robustly around 0, since the difference $\alpha_i - \alpha_{i+q}$ is both negative and positive, in practice.

A significant reduction of computational time can thus be attained by selecting a low number of factors d. A consequence of a low number of factors is that the instantaneous correlation matrix (ρ_{ij}) cannot be exactly fit to a given general correlation matrix. The procedure for fitting a generic market model to correlation is exactly the same as for the LIBOR market model. For fitting a low-factor LIBOR market model to correlation, the reader is referred to Pietersz & Groenen (2004), Grubišić & Pietersz (2005), Wu (2003) and Rebonato (2002).

Algorithm 2 An $\mathcal{O}(nd)$-algorithm for approximately calculating the forward swap rates for a time step in the CMS(q) market model (exact when $q = n$), under the terminal measure. The number of factors is denoted by d. The log forward rates, $\log \mathbf{f}(t) = (\log f_{i(t):e(i(t))}(t), \ldots, \log f_{n:e(n)}(t))$ at time t, and $\log \mathbf{f}(t + \Delta t)$ at time $t + \Delta t$, are denoted by $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$, respectively. Here $\boldsymbol{\Sigma} = (\sigma_{ij})$ governs the volatility, with σ_{ij} the time-t volatility of forward rate $f_{i:e(i)}$ with respect to factor j. Here, $e(\cdot)$ is defined in (2). $\Delta \mathbf{w}$ should be sampled from a $\mathcal{N}(\mathbf{0}, \sqrt{\Delta t}\boldsymbol{I}_d)$ distribution.

Input: n; d, q $(1 \le d, q \le n)$; $\boldsymbol{\phi}^{(1)}, \boldsymbol{\alpha} \in \mathbb{R}^n$; $\Delta \mathbf{w} \in \mathbb{R}^d$; $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times d}$; Δt.
Output: $\boldsymbol{\phi}^{(2)} \in \mathbb{R}^n$.

$\beta_{n+1} \Leftarrow 1$. $\varpi_{n+1} \Leftarrow 0$.
for $i = n, \ldots, i(t)$ **do**
  $\varpi_i \Leftarrow \varpi_{i+1} + \alpha_i \beta_{i+1} - 1_{\{i<n \text{ \& } e(i)=e(i+1)-1\}} \alpha_{e(i+1)-1} \beta_{e(i+1)}$.
  $f_i^{(1)} \Leftarrow \exp(\phi_i^{(1)})$.
  $\beta_i \Leftarrow \varpi_i f_i^{(1)} + \beta_{e(i)}$.
  If $i = n$, set $\mathbf{v}_n \Leftarrow \mathbf{0} \in \mathbb{R}^d$, else $(i < n)$, set

$$\mathbf{v}_i \Leftarrow \big(1 + \alpha_i f_{i+1}^{(1)}\big)\mathbf{v}_{i+1} + \alpha_i(\beta_{i+1} - \beta_{e(i+1)})\boldsymbol{\sigma}_{i+1}.$$

  $\phi_i^{(2)} \Leftarrow \phi_i^{(1)} + (-\frac{1}{\varpi_i}\mathbf{v}_i - \frac{1}{2}\boldsymbol{\sigma}_i) \cdot \boldsymbol{\sigma}_i \Delta t + \boldsymbol{\sigma}_i \cdot \Delta \mathbf{w}$.
end for

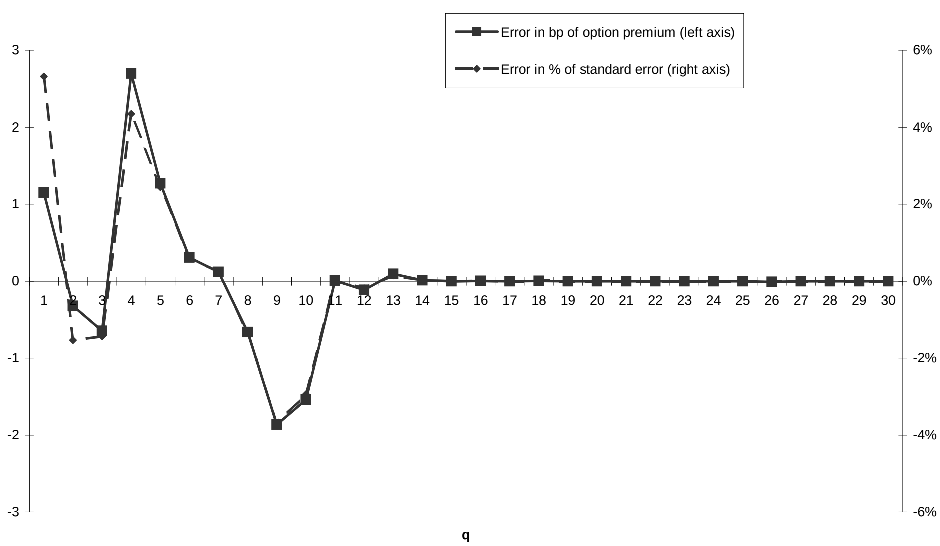

Figure 4: Test results of exact versus approximate drifts in CMS(q) models.

6 Generic calibration to correlation

When each interest rate derivative has its own generic market model that is used for its valuation and risk management, then the associated input correlation to those models involves different interest rates. There is a relationship between these correlations, which allows for netting correlation risk. Moreover, utilizing the relationship between correlations means that correlation is determined consistently across different products. In general all interest rate correlations stem from correlations between different segments of the yield curve. We show how instantaneous forward LIBOR correlations can be used to determine subsequently the correlations for any generic market model. A further advantage is that only forward LIBOR correlations have to be determined and administered.

The key to the method is the well-known fact that, within the LIBOR market model, the instantaneous volatility vector $\boldsymbol{\sigma}_{s:e}(t)$ of a forward swap rate $f_{s:e}$ can be approximated as weighted averages of instantaneous volatility vectors $\boldsymbol{\sigma}_i(t)$ of forward LIBORs. If we denote the approximation by $\tilde{\boldsymbol{\sigma}}_{s:e}(t)$, then:

$$\tilde{\boldsymbol{\sigma}}_{s:e}(t) = \sum_{i=s}^{e-1} w_i^{s:e}(0)\boldsymbol{\sigma}_i(t).$$

An expression for the weights $w_i^{s:e}$ may be found, e.g., in Hull & White (2000, page 53). Instantaneous forward rate correlations $\rho_{s(1):e(1),s(2):e(2)}(t)$ can thus be expressed as a function of instantaneous forward LIBOR correlations $\rho_{ij}(t)$,

$$\rho_{s(1):e(1),s(2):e(2)}(t) = \frac{\boldsymbol{\sigma}^T_{s(1):e(1)}(t)\boldsymbol{\sigma}_{s(2):e(2)}(t)}{\sqrt{\boldsymbol{\sigma}^T_{s(1):e(1)}(t)\boldsymbol{\sigma}_{s(1):e(1)}(t)\boldsymbol{\sigma}^T_{s(2):e(2)}(t)\boldsymbol{\sigma}_{s(2):e(2)}(t)}},$$

where $\boldsymbol{\sigma}^T_{i:j}(t)\boldsymbol{\sigma}_{k:l}(t)$ can be approximated by $\tilde{\boldsymbol{\sigma}}^T_{i:j}(t)\tilde{\boldsymbol{\sigma}}_{k:l}(t)$, with

$$\tilde{\boldsymbol{\sigma}}^T_{i:j}(t)\tilde{\boldsymbol{\sigma}}_{k:l}(t) = \sum_{m_1=i}^{j-1}\sum_{m_2=k}^{l-1} w^{i:j}_{m_1}(0)w^{k:l}_{m_2}(0)|\boldsymbol{\sigma}_{m_1}(t)||\boldsymbol{\sigma}_{m_2}(t)|\rho_{m_1m_2}(t).$$

7 Conclusions

A generalization of market models has been studied, whereby arbitrary forward rates are allowed to span a tenor structure. The benefit of such generalization is that straightforward volatility-calibration can be achieved for the fixings of LIBOR or swap rates relevant to an interest rate derivative. Generic market models are therefore particularly apt for pricing and risk management of CMS and hybrid coupon swaps, and callable and cancellable versions thereof, in particular, Bermudan CMS swaptions and fixed-maturity Bermudan swaptions. We showed that the LIBOR and swap market models are special cases of the generic market model framework. Necessary and sufficient conditions were derived for a set of forward swap agreements to provide a unique solution for discount bond prices, essentially regardless of the scenario of attained forward rates. The major novelty of this paper is the derivation of generic expressions for no-arbitrage drift terms in generic market models.

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A Rationale for approximation 1

We proceed by induction on $i = k-1, \ldots, i(t)$.

- For $i = k-1$: $\tilde{p}^{(k)}_{i:\mu(i,k)+1} = \tilde{p}^{(k)}_{k-1:\min(k,k-1+q)} = \alpha_{k-1}\tilde{b}^{(k)}_k = \alpha_{k-1}$.
- For $i = k-2, \ldots, k-q$, we have $\min(k, i+q) = k$. The quantity $\tilde{b}^{(k)}_{i+1}$ satisfies:

$$\tilde{b}^{(k)}_{i+1} = f_{i+1:e(i+1)}\tilde{p}^{(k)}_{i+1:k}. \tag{26}$$

  To see this, note that from line 3 of Algorithm 1, we have:

$$\tilde{b}^{(k)}_{i+1} = f_{i+1:e(i+1)}\tilde{p}^{(k)}_{i+1:i+q+1} + \tilde{b}^{(k)}_{i+q+1}. \tag{27}$$

From the definition $\tilde{b}_j^{(k)} = (\mathbf{U}^{-1}\mathbf{c}_k)_j$ in (14), we deduce that $\tilde{b}_j^{(k)} = 0$ for $j > k$, from which (26) follows. We obtain:

$$\begin{aligned}
\tilde{p}^{(k)}_{i:\mu(i,k)+1} &= \tilde{p}^{(k)}_{i:k} = \tilde{p}^{(k)}_{i+1:k} + \alpha_i \tilde{b}^{(k)}_{i+1} = \tilde{p}^{(k)}_{i+1:k}\Big(1 + \alpha_i f_{i+1:e(i+1)}\Big) \\
&\overset{(*)}{=} \alpha_{k-1} \prod_{m=i+1}^{k-2} \Big(1 + \alpha_m f_{m+1:e(m+1)} \Big)\Big(1 + \alpha_i f_{i+1:e(i+1)}\Big) \\
&= \alpha_{k-1} \prod_{m=i}^{k-2} \Big(1 + \alpha_m f_{m+1:e(m+1)} \Big),
\end{aligned}$$

where equality $(*)$ follows from the induction hypothesis.

- For $i = k-q-1, \ldots, i(t)$, we have $\min(k, i+q) = i+q$. From (27), we deduce:

$$\begin{aligned}
\tilde{p}^{(k)}_{i:\mu(i,k)+1} &= \tilde{p}^{(k)}_{i:i+q} = \alpha_i \tilde{b}^{(k)}_{i+1} - \alpha_{i+q}\tilde{b}^{(k)}_{i+q+1} + \tilde{p}^{(k)}_{i+1:i+q+1} \\
&= \alpha_i \Big(f_{i+1:e(i+1)} \tilde{p}^{(k)}_{i+1:i+q+1} + \tilde{b}^{(k)}_{i+q+1} \Big) \\
&\qquad - \alpha_{i+q}\tilde{b}^{(k)}_{i+q+1} + \tilde{p}^{(k)}_{i+1:i+q+1} \\
&\overset{(*)}{\approx} \tilde{p}^{(k)}_{i+1:i+q+1}\Big(1 + \alpha_i f_{i+1:e(i+1)}\Big) \\
&= \alpha_{k-1} \prod_{m=i}^{k-2} \Big(1 + \alpha_m f_{m+1:e(m+1)} \Big),
\end{aligned}$$

where in approximation $(*)$, we have used $\alpha_i \approx \alpha_{i+q}$. □

B Proof of Lemma 2

$$\begin{aligned}
\mathbf{v}_i &= \sum_{k=i+1}^{n} \Big(\hat{b}_k - \hat{b}_{e(k)}\Big)\alpha_{k-1} \prod_{m=i}^{k-2} \Big(1 + \alpha_m f_{m+1:e(m+1)}\Big)\boldsymbol{\sigma}_{k:e(k)} \\
&= \Big(\hat{b}_{i+1} - \hat{b}_{e(i+1)}\Big)\alpha_i \boldsymbol{\sigma}_{i+1:e(i+1)} + \Big(1 + \alpha_i f_{i+1:e(i+1)}\Big) \times \\
&\qquad \left\{ \sum_{k=i+2}^{n} \Big(\hat{b}_k - \hat{b}_{e(k)}\Big)\alpha_{k-1} \prod_{m=i+1}^{k-2} \Big(1 + \alpha_m f_{m+1:e(m+1)}\Big)\boldsymbol{\sigma}_{k:e(k)} \right\} \\
&= \Big(1 + \alpha_i f_{i+1:e(i+1)}\Big)\mathbf{v}_{i+1} + \alpha_i \Big(\hat{b}_{i+1} - \hat{b}_{e(i+1)}\Big)\sigma_{i+1:e(i+1)},
\end{aligned}$$

for $i < n$, which was to be shown. □