Swap vega in BGM: pitfalls and alternatives

Practitioners who are developing the Libor BGM model for risk management of a swap-based interest rate derivative be warned: for certain volatility functions the estimate of swap vega may be poor. This may occur for time-homogeneous forward or swap rate volatility, but it does not occur for constant volatility. Raoul Pietersz and Antoon Pelsser explain and present an alternative method that estimates vega with clarity at a low number of simulation paths for all volatility functions

The Libor BGM interest rate model introduced by Brace, Gatarek & Musiela (1997) and others is one of the most popular such models among both academics and practitioners alike. One reason is that it has the potential of risk managing interest rate derivatives that depend on both the cap and swaption markets, which would render BGM as a central interest rate model. It features lognormal Libor rates and almost lognormal swap rates and consequently also the market standard Black formula for caps and swaptions.

The volatility function allows for future volatility modelling. In this article, however, we show that this introduces a pitfall when calculating swap vega. The swap vega is the sensitivity of a derivative with respect to the volatility of an underlying swaption. In combination with certain volatility functions, BGM may produce poorly estimated swap vega when these are calculated by re-calibration and with a low number of simulation paths, say 10,000. We have seen this occur for time-homogeneous forward or swap rate volatility, but it does not occur for constant volatility. Incorrect swap vegas leave practitioners unknowingly taking on large uncovered positions and thus increase the variance of profit and loss. Unstable vegas lead to large and unnecessary transaction costs when rebalancing the hedging portfolio based on fluctuations of vega that are not really material.

Re-calibration approach

A common and usually very successful method for calculating a Greek in a model equipped with a calibration algorithm is to perturb market input, re-calibrate and then revalue the option. The difference in value divided by the perturbation size is then an estimate for the Greek. If, however, this technique is applied to the calculation of swap vega in the Libor BGM model, then it may yield estimates with large uncertainty, that is, the standard error of the vega is relatively high. We have observed this phenomenon for time-homogeneous forward or swap rate volatility, but not for constant volatility. Evidently, the uncertainty disappears by increasing the number of simulation paths, but the number required for clarity (that is, an acceptable standard error) can by far exceed 10,000, which is probably the maximum in a practical environment. The phenomenon is illustrated in the next section.

Examples of swap vega based on re-calibration

Take a tenor structure $0 =: T_0, T_1, \dots, T_{N+1}$ with corresponding forward Libor rates L_i for borrowing/lending over the period $[T_i, T_{i+1}]$. Each forward rate is modelled as:

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t)\,dW^{(i+1)}(t)$$

Here, σ_i denotes forward rate volatility and $W^{(i+1)}$ denotes a Brownian motion under the ith forward measure. The correlation structure is modelled by:

$$dW^{(i+1)}(t)\,dW^{(j+1)}(t) = \rho_{ij}(t)$$

Swap rate instantaneous volatility is almost deterministic and its deterministic approximate is denoted by $\sigma_{i:j}(t)$ for a swap starting at T_i and ending at T_{j+1}.

Three volatility calibrations are considered:

- **Time-homogeneous forward rate volatility (THFRV).** Here, the parameters are γ_j, $j = 1, \dots, N$, denoting the volatility when the index to maturity is j, that is, $\sigma_i(t) = \gamma_j$, $T_{i-j} \le t < T_{i-j+1}$. The calibration algorithm is based on Newton Rhapson. The future cap volatility curve is maintained.
- **Time-homogeneous swap rate volatility (THSRV).** Here, the swap rate instantaneous volatility $\sigma_{i:j}(t)$ is assumed time-homogeneous. The calibration algorithm is a two-stage bootstrap algorithm. The first and second stages are described in, for example, equation (6.20) and section 7.4, respectively, of Brigo & Mercurio (2001). The future swaption volatility curve is maintained.
- **Constant forward rate volatility (CONST).** Here, $\sigma_i(t) = (const)_i$. This implies constant swap rate volatility and distorts future volatility curves.

The forward rate correlation was calibrated by means of a principal component analysis (Hull & White, 2000).

We considered a 31NC1 co-terminal Bermudan payer's swaption deal struck at 5% with annual compounding. The notation *xNCy* denotes an '*x* non-call *y*' Bermudan option, which is exercisable into a swap with a maturity of *x* years from today and callable only after *y* years. The option is callable annually. The BGM tenor structure is $0 < 1 < 2 < \dots < 31$. All initial forward rates are taken to equal 5%. The time-zero forward rate instantaneous correlation is assumed given by the form:

$$\rho_{ij}(0) = \exp\left\{-\beta\left|T_i - T_j\right|\right\}, \quad \beta = 5\%$$

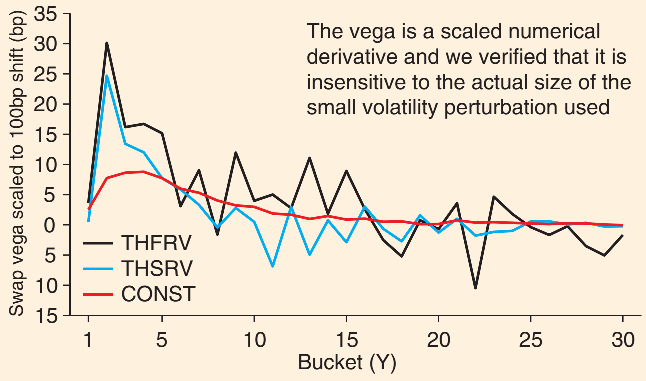
The market European-style swaption volatilities are as shown in table A. The numerical results are shown in figures 1 and 2.

The vegas are poorly estimated for the THFRV and THSRV cases, whereas the vega are more accurately estimated by the constant volatility calibration. For the THFRV case, the vegas have been calculated at 1 million simulation paths (see figure 3). We see that, for a constant volatility cali-

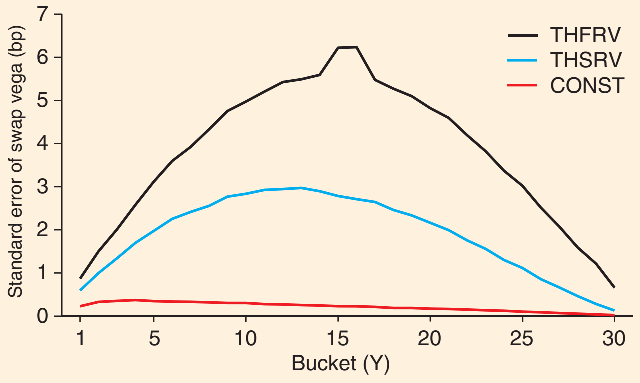
A. Swaption volatilities for re-calibration illustration

Expiry	1	2	3	...	28	29	30
Tenor	30	29	28	...	3	2	1
Swaption volatility	15.0%	15.2%	15.4%	...	20.4%	20.6%	20.8%

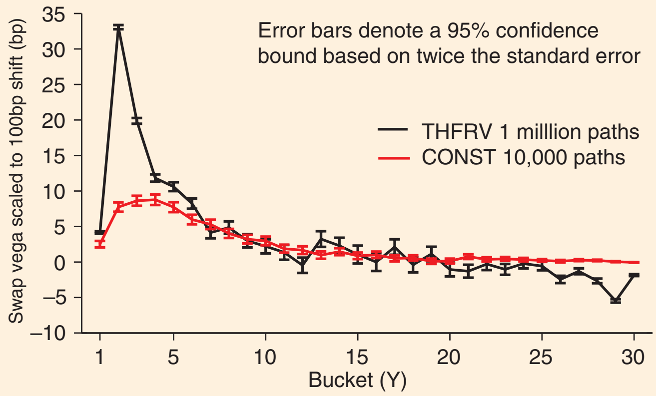
1. Re-calibration swap vega results for 10,000 simulation paths



2. Empirical standard errors of the vega for 10,000 simulation paths



3. Re-calibration THFRV vega results for 1 million simulation paths

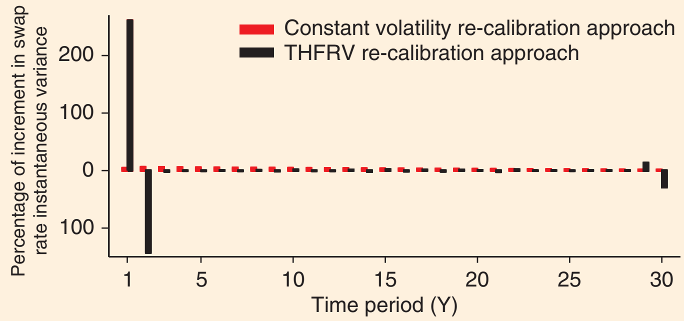


bration, the vega is estimated with low uncertainty. The number of simulation paths needed for clarity of vega thus depends heavily on the chosen calibration.

An explanation

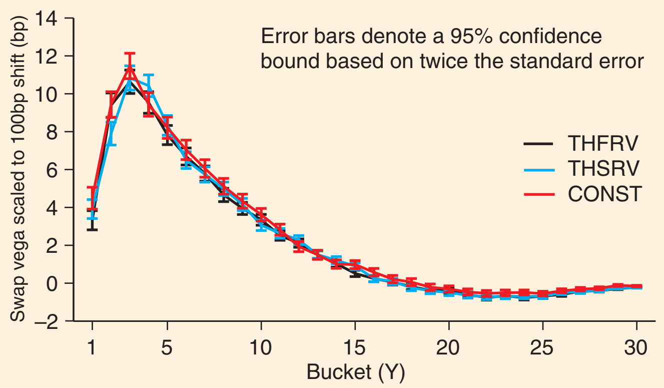
The key to the explanation of the poor estimate of vega is the change in swap rate instantaneous variance. We observed this change and noted that, for the THFRV and THSRV re-calibration approaches, the instantaneous variance increment (in the limit) is completely different from a constant volatility increment (see figure 4).

4. Observed change in swap rate instantaneous variance



The observed change in swap rate instantaneous variance for the THFRV and CONST re-calibration approach for the deal set-up referred to in the text. The concern here is the calculation of swap vega corresponding to bucket 30. To accomplish this, the price differential has to be calculated in the limit of 30 × 1 swaption implied volatility perturbation $\Delta\sigma$ tending to zero. This implies an instantaneous variance of $30\Delta\sigma^2$. The total variance increment has to be distributed over all time periods. Note that for both data sets the sum of the variance increments equals 100%

5. Vega calculated with the alternative approach



To understand that the resulting vega is more difficult to estimate for the THFRV/THSRV case, note that the vega is a multiple of an expectation of a difference in discounted payouts in a model with either perturbed volatility or the original volatility:

$$vega_{i:N} = cE\left[P_{i:N} - P\right]$$

Here, $vega_{i:N}$ denotes the estimated vega for bucket i, c is the reciprocal of the perturbation size $\Delta\sigma_{i:N}$, and $P_{i:N}$ and P denote the discounted payout in the perturbed and original model, respectively. The expectation under the risk-neutral measure is denoted by $E[\cdot]$. The simulation variance of the vega is thus given by:

$$Var\left[vega_{i:N}\right] = c^2 Var\left[P_{i:N} - P\right] = c^2\left\{Var\left[P_{i:N}\right] - 2Cov\left[P_{i:N}, P\right] + Var\left[P\right]\right\}$$

The vega standard error is thus minimised if the covariance between the discounted payout in either the original and the perturbed model is largest. This occurs under small perturbations of volatility as implied by the constant volatility regime. In the presence of a perturbation such as the THFRV re-calibration, however, the stochasticity in the simulation is basically moved around to other time periods (in our case from period two to one), but increments over different periods are independent, so the covariance is decreased. This leads to a larger uncertainty in the vega.

In the following, an alternative method is presented that estimates swap vega with a low standard error for any volatility function.

Swap vega and the swap market model

A swap market model features lognormally distributed swap rates. The implied swaption volatility in terms of instantaneous volatility is given by:

$$\sigma_{Implied} = \sqrt{\frac{1}{T}\int_0^T \left|\sigma(s)\right|^2 ds} \tag{1}$$

The swap vega is defined as the sensitivity of the value V of a derivative with respect to the implied swaption volatility:

$$vega = \frac{\partial V}{\partial \sigma_{Implied}}$$

As may be seen from equation (1), there are an uncountable number of perturbations in the swap rate instantaneous volatility to obtain the very same perturbation in the Black implied swaption volatility. There is, however, a natural one-dimensional parameterised perturbation, that is, a simple proportional increment. In equational terms:

$$\left\{\sigma(\ )\right\}_{perturbed} = (1+\varepsilon)\sigma(\ ) \tag{2}$$

It may be shown that the above perturbation leads to:

☐ An increase in the implied volatility for the relevant swaption bucket.
☐ All other swaption volatilities remaining unchanged.
☐ The swap rate correlation remaining unchanged.

An alternative method

Here, we present and test a simple alternative method for calculating the vega. This method was developed by Rebonato (2002) in terms of covariance matrices, and independently by Pietersz & Pelsser (2004) in terms of the volatility vectors.

The method is based on a perturbation in the forward rate volatility to match a constant swap rate volatility increment. Rebonato (1999) showed that the swap rate volatility vector is a weighted average of forward rate volatility vectors:

$$\bar{\sigma}_{i:N}(t) = \sum_{j=1}^{N} w_j^{i:N}(t)\bar{\sigma}_j(t)$$

and that an adequate approximating formula for European-style swaption prices can be obtained by evaluating the weights at time zero. Though we have used the latter approximation in our calculations, we mention here that Jäckel & Rebonato (2003) have developed a more accurate approximation for the weights, by means of the so-called shape correction. This more general approximation readily extends as well to the technique presented hereafter, as mentioned in Rebonato (2002). Write $w_j^{i:N} := w_j^{i:N}(0)$ and establish the convention that $\bar{\sigma}_i(t) = \bar{\sigma}_{i:N}(t) = 0$, $t > T_i$, then the volatility vectors can be jointly related through the matrix equation:

$$\left[\bar{\sigma}_{\cdot:N}(t)\right] = W\left[\bar{\sigma}_{\cdot}(t)\right]$$

The perturbation in swap rate volatility for the kth bucket prescribed by equation (2) is:

$$\left[\bar{\sigma}_{\cdot:N}(t)\right] \to \left[\bar{\sigma}_{\cdot:N}(t)\right] + \varepsilon\left[0\cdots0\ \bar{\sigma}_{k:N}(t)\ 0\cdots0\right]^T$$

The corresponding perturbation in the Libor volatility vectors is given by:

$$\left[\bar{\sigma}_{\cdot}(t)\right] \to \left[\bar{\sigma}_{\cdot}(t)\right] + \varepsilon W^{-1}\left[0\cdots0\ \bar{\sigma}_{k:N}(t)\ 0\cdots0\right]^T$$

With the new Libor volatility vectors, prices can be recalculated in the BGM model and the vegas calculated. The vegas calculated with the alternative approach are shown in figure 5. Interestingly, the vegas are now more or less equal, whereas the prices are not. A possible explanation could be the imposed identical volatility perturbation.

Conclusions

We showed that care should be taken when calculating swap vega per bucket in the Libor BGM model by re-calibration, because the perturbation in instantaneous swap rate volatility is hidden and potentially unstable. However, if the method proposed in this article is applied, it is possible to obtain correct swap vega per bucket in the BGM model for any volatility function at a low number of simulation paths. ■

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