Bridging Brownian LIBOR

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Prices of interest rate derivatives in the LIBOR market model can be efficiently approximated by drift approximations say **Raoul Pietersz**, **Antoon Pelsser and Marcel van Regenmortel**. They introduce a novel drift approximation based on the Brownian bridge.

Introduction

Recent developments in the interest rate derivatives industry have seen an increase in complexity of products. European call and put options on LIBOR and swaps, so-called *caps/floors* and *swaptions*, once deemed exotic, are now considered plain vanilla. The newer more complex products, named *exotic LIBOR derivatives*, are tailored to investor requirements and contain many additional features. For example, these products may be callable at multiple exercise dates, or may contain knock-out or knock-in features. Also, a 'fixed' or 'floating' payment in the underlying *exotic* swap, depending on realized LIBOR ℓ, may be, for example, capped or floored, $\min(\ell, k)$ or $\max(\ell, k)$, with k the maximum or minimum coupon rate; inverse, $k - \ell$; leveraged, $\lambda \times \ell$, with λ the leverage rate, or; ranged, $r \times \ell$, with r the ratio of the number of days over the accrual period for which a certain reference rate is within a contractually prescribed range. This list of features is certainly not exhaustive, and moreover combinations are possible, also the type and coupon rates can vary with the index of the payment.

Caps and swaptions are valued with the Black (1976) formula. For exotic LIBOR derivatives however, the now more or less market standard for valuation and risk management thereof is the LIBOR market model of Brace, Gątarek & Musiela (1997). Prices within the LIBOR market model are invariably calculated by Monte Carlo (MC) simulation. The drawback of MC is the relatively long computational time. The time involved is multiplied by the amount of derivatives in the book and the number of required risk sensitivities. Usually, there is a sensitivity per each underlying LIBOR or swap rate used to build the yield curve, and per each volatility of underlying cap or swaption to which the model has been calibrated. As the size of the book grows, the lengthy computational time can start to exceed an overnight run, and can thus potentially hinder effective risk management.

In this paper, we examine drift approximations that render the state of the LIBOR model as Markovian with respect to a low number of Wiener processes. Hunter, Jäckel & Joshi (2001) and Kurbanmuradov, Sabelfeld & Schoenmakers (2002) introduced drift approximations for the LIBOR market model, in order to limit MC computational time. Together with a non-restrictive assumption on the volatility, named *separability*, single time-step drift approximations enable the implementation of *finite differences* or *partial differential equations (PDEs)*, that are much more efficient than MC. Single time-step pricing is accurate for deals up to, say, 10 years. We propose a novel drift approximation based on the Brownian bridge. The novel Brownian bridge scheme has least-squares error over a certain class (to be defined) of single time-step discretizations. Viewed as a MC discretization, the Brownian bridge scheme converges weakly with order 1. The presentation in this paper is based on the research article of Pietersz, Pelsser & Van Regenmortel (2004).

Drift approximations, thus also the Brownian bridge version, are employed in two other areas. First, Piterbarg (2003, Section 13) uses drift approximated prices as control variates. Second, the implementation of the likelihood ratio method (LRM), for efficient calculations of risk sensitivities, requires the forward rates to be Gaussian, and thus drift approximations can be used, as proposed by Glasserman & Zhao (1999). For an exposition of LRM, see Jäckel (2003).

The outline of this paper is as follows. First, we show how a single time-step discretization in combination with separability leads to finite difference pricing. Second, the Brownian bridge discretization is introduced, and other single time-step discretizations are mentioned. Third and fourth, the Brownian bridge scheme is discussed for single and multi time-steps, respectively. Fifth, the framework is illustrated with a 2-factor model. Sixth, we end with conclusions.

Single Time-step Pricing Framework

Let $0 =: t_0 < t_1 < \ldots < t_{n+1}$ denote a tenor structure. Associated with this tenor structure are forward LIBOR rates f_i, $i = 1, \ldots, n$, with f_i corresponding to the period $[t_i, t_{i+1}]$ with day count fraction δ_i. The log[1] forward rates satisfy the stochastic differential equation (SDE)

$$d\log f_i = \{\mu_i(\mathbf{f}, t) - \tfrac{1}{2}||\boldsymbol{\sigma}_i(t)||^2\}dt + \boldsymbol{\sigma}_i(t) \cdot d\mathbf{w}. \tag{1}$$

Here, $\boldsymbol{\sigma}_i(t)$ denotes a d-tuple, with entry k relating to the volatility with respect to the k^{th} Brownian motion w_k driving the model. We may define the absolute level of instantaneous volatility by $\sigma_i(t) = ||\boldsymbol{\sigma}_i(t)||$ and the instantaneous correlation $\rho_{ij}(t)$ by $\boldsymbol{\sigma}_i(t) \cdot \boldsymbol{\sigma}_j(t) = \sigma_i(t)\sigma_j(t)\rho_{ij}(t)$. The term $\mu_i(\mathbf{f}, t)$ is determined by the chosen measure. In this paper, we work only[2] with the terminal measure, associated with the discount bond maturing at time t_{n+1} as numeraire. For the terminal measure, we have

$$\mu_i(\mathbf{f}, t) = -\sum_{j=i+1}^{n} \frac{\delta_j f_j}{1+\delta_j f_j}\sigma_i(t)\sigma_j(t)\rho_{ij}(t) = -\sum_{j=i+1}^{n} \frac{\delta_j f_j \boldsymbol{\sigma}_i(t) \cdot \boldsymbol{\sigma}_j(t)}{1+\delta_j f_j}. \tag{2}$$

Suppose $\tau_1 < \ldots < \tau_M$ is a time discretization. Denote $z_i(u, v) = \int_u^v \boldsymbol{\sigma}_i(s) \cdot d\mathbf{w}(s)$. We consider discretizations of the form

$$\begin{aligned} \log f_i(\tau_{m+1}) &= \log f_i(\tau_m) + \bar{\mu}_i(\tau_m, \tau_{m+1}, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})) \\ &\quad - \frac{1}{2}\int_{\tau_m}^{\tau_{m+1}} \sigma_i^2(s)\, ds + z_i(\tau_m, \tau_{m+1}). \end{aligned} \tag{3}$$

Here $\bar{\mu}$ stems from the discretization scheme applied, which can be the Euler, predictor-corrector or Brownian bridge schemes. Details of those schemes are given in the next section. The single time-step forward rates process $\mathbf{f}^{\text{DA}}$ is then defined by

$$f_i^{\text{DA}}(t) = f_i(0)\exp\left\{\bar{\mu}_i(0, t, \mathbf{f}(0), \mathbf{z}(0, t)) - \frac{1}{2}\int_0^t \sigma_i^2(s)\, ds + z_i(0, t)\right\}. \tag{4}$$

('DA' for drift approximated.) Now suppose that we make the following separability assumption on volatility,

$$\boldsymbol{\sigma}_i(t) = \boldsymbol{\gamma}(t)\mathbf{v}_i, \text{(entry-by-entry multiplication)}, \tag{5}$$

for some d-dimensional vector-valued $\boldsymbol{\gamma}(\cdot)$ and $\mathbf{v}_i$. Then the state of the n-dimensional drift approximated forward rates process is fully determined by the state of the d-dimensional process $\mathbf{x}(0, \cdot)$, defined by,

$$\mathbf{x}(0, t) = \int_0^t \boldsymbol{\gamma}(s)d\mathbf{w}(s), \text{(entry-by-entry multiplication)}. \tag{6}$$

This can be seen as follows. We have

$$\begin{aligned} z_i(0, t) &= \int_0^t \boldsymbol{\sigma}_i(s) \cdot d\mathbf{w}(s) = \int_0^t \mathbf{v}_i \boldsymbol{\gamma}(s) \cdot d\mathbf{w}(s) \\ &= \mathbf{v}_i \cdot \int_0^t \boldsymbol{\gamma}(s)d\mathbf{w}(s) = \mathbf{v}_i \cdot \mathbf{x}(0, t), \end{aligned} \tag{7}$$

and then substituting (7) into (4) yields the desired result. This d-dimensional representability result enables pricing by a d-dimensional PDE, instead of a n-dimensional. The d-dimensional PDE is

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sum_{k=1}^{d} \gamma_k^2(t)\frac{\partial^2 \pi}{\partial x_k^2} = 0, \tag{8}$$

with appropriate boundary conditions. Here π denotes the numeraire-relative value of the derivative contract. The reduction of dimensionality ('*rank*') from n to d highlights the importance of rank reduction of correlation matrices. For a recent review article on this topic, see Pietersz & Groenen (2004).

The Brownian Bridge LIBOR Drift Approximation

In this section, first, we mention other available discretizations. Second, we present the new Brownian bridge LIBOR scheme.

The schemes that we mention are Euler, predictor-corrector and Milstein. We only mention the form of $\bar{\mu}$ in (3), since the other scheme-components in (3) are equal for all discretizations considered. For Euler,

$$\begin{aligned} \bar{\mu}_i^{\text{Euler}}(\tau_m, \tau_{m+1}, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})) &= (\tau_{m+1} - \tau_m)\times \\ &\quad -\left\{\sum_{j=i+1}^{n} \frac{\delta_j f_j(\tau_m)\boldsymbol{\sigma}_i(\tau_m) \cdot \boldsymbol{\sigma}_j(\tau_m)}{1+\delta_j f_j(\tau_m)}\right\}. \end{aligned} \tag{9}$$

The predictor-corrector scheme, introduced to a finance setting by Hunter et al. (2001), is as

$$\begin{aligned} \bar{\mu}_i^{\text{predictor-corrector}}&(\tau_m, \tau_{m+1}, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})) = (\tau_{m+1} - \tau_m)\times \\ & -\left\{\frac{1}{2}\sum_{j=i+1}^{n} \frac{\delta_j f_j(\tau_m)\boldsymbol{\sigma}_i(\tau_m) \cdot \boldsymbol{\sigma}_j(\tau_m)}{1+\delta_j f_j(\tau_m)}\right. \\ & \left. +\frac{1}{2}\sum_{j=i+1}^{n} \frac{\delta_j f_j(\tau_{m+1})\boldsymbol{\sigma}_i(\tau_{m+1}) \cdot \boldsymbol{\sigma}_j(\tau_{m+1})}{1+\delta_j f_j(\tau_{m+1})}\right\}. \end{aligned} \tag{10}$$

Note that scheme (10) is calculated iteratively for $i = n, n-1, \ldots$, since the expression for $\bar{\mu}_i$ involves the time-τ_{m+1} forward rates $f_j(\tau_{m+1})$, $j = i+1, \ldots, n$.

The Milstein scheme is a *second order* scheme, which means that the error of the discretization diminishes quadratically with the time-step $\Delta\tau \downarrow 0$. Intuitively, one can anticipate that the error then also *grows* quadratically with *increasing* $\Delta\tau$, which implies that Milstein is not efficient for the single time-step pricing framework. Indeed, in the tests of the next section, it is shown that the Milstein scheme performs worst over all schemes considered in this paper. We therefore omit further discussion of the Milstein scheme.

The idea of the Brownian bridge scheme is to incorporate *all* available information in the drift-estimate given the Brownian increment. In mathematical terms, this amounts to taking the expectation of the drift, conditional on the Brownian increment. In terms of a formula,

$$\begin{aligned}&\bar{\mu}_i^{\text{Browne-bridge}}(\tau_m, \tau_{m+1}, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1}))\\ &= -E\left[\int_{\tau_m}^{\tau_{m+1}} \sum_{j=i+1}^{n} \frac{\delta_j f_j(s)\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j f_j(s)}\,\mathrm{d}s \,\middle|\, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})\right].\end{aligned} \tag{11}$$

That is the definition of the Brownian bridge scheme and there you have it. All things easier said than done, the question still remains of how to numerically approximate the expression in (11). This approximation is detailed in the following 4 steps.

1. Interchange the expectation and the integration over time; this is allowed by application of Fubini's theorem,

$$\begin{aligned}&E\left[\int_{\tau_m}^{\tau_{m+1}} \sum_{j=i+1}^{n} \frac{\delta_j f_j(s)\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j f_j(s)}\,\mathrm{d}s \,\middle|\, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})\right]\\ &= \int_{\tau_m}^{\tau_{m+1}} E\left[\sum_{j=i+1}^{n} \frac{\delta_j f_j(s)\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j f_j(s)} \,\middle|\, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})\right]\mathrm{d}s.\end{aligned} \tag{12}$$

2. (*Approximation*) Assume, for purpose of approximating (12) only, that the forward rate $f_j(s)$ has deterministic drift.

$$\begin{aligned}&\int_{\tau_m}^{\tau_{m+1}} E\left[\sum_{j=i+1}^{n} \frac{\delta_j f_j(s)\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j f_j(s)} \,\middle|\, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})\right]\mathrm{d}s \approx\\ &\int_{\tau_m}^{\tau_{m+1}} E\left[\sum_{j=i+1}^{n} \frac{\delta_j f_j^{\text{DD}}(s)\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j f_j^{\text{DD}}(s)} \,\middle|\, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})\right]\mathrm{d}s\end{aligned} \tag{13}$$

('DD' for 'deterministic drift'.) Having assumed (13), $f_j^{\text{DD}}(s)|\{f_j(\tau_m), f_j(\tau_{m+1})\}$ becomes distributed as a geometric Brownian bridge. It is a well-known fact for the Brownian bridge that the drift prior to conditioning is irrelevant to the process after conditioning. Because of this nil-dependence of the drift, it follows that the assumption of deterministic drift instead of the stochastic drift (2) probably has a negligible impact on the drift approximation. Indeed, in the tests of the next section, the Brownian bridge scheme is shown to have very high accuracy. We stress here once more that the assumption of deterministic drift is used only for approximation from (12) to (13), and not in the discretization scheme (3) itself.

3. The mean of the Brownian bridge forward rate is given by (see Appendix A of Pietersz et al. (2004)),

$$\begin{aligned}&E\lfloor f_j^{\text{DD}}(t)|f_j(\tau_m), f_j(\tau_{m+1})\rfloor = f_j(\tau_m)\left(\frac{f_j(\tau_{m+1})}{f_j(\tau_m)}\right)^{\frac{\alpha_j^2(\tau_m,t)}{\alpha_j^2(\tau_m,\tau_{m+1})}}\\ &\times \exp\left\{\frac{1}{2}\frac{\alpha_j^2(\tau_m,t)}{\alpha_j^2(\tau_m,\tau_{m+1})}(\alpha_j^2(\tau_m,\tau_{m+1}) - \alpha_j^2(\tau_m,t))\right\},\end{aligned} \tag{14}$$

with $\alpha_j^2(u,v) = \int_u^v \sigma_j^2(s)\,\mathrm{d}s$.

4. (*Approximation*) For calculation of the expectation in (13), substitute the forward LIBOR rates with their means,

$$\begin{aligned}&\int_{\tau_m}^{\tau_{m+1}} E\left[\sum_{j=i+1}^{n} \frac{\delta_j f_j^{\text{DD}}(s)\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j f_j^{\text{DD}}(s)} \,\middle|\, \mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})\right]\mathrm{d}s \approx\\ &\int_{\tau_m}^{\tau_{m+1}} \sum_{j=i+1}^{n} \frac{\delta_j E[f_j^{\text{DD}}(s)|\mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})]\boldsymbol{\sigma}_i(s)\cdot\boldsymbol{\sigma}_j(s)}{1+\delta_j E[f_j^{\text{DD}}(s)|\mathbf{f}(\tau_m), \mathbf{z}(\tau_m, \tau_{m+1})]}\,\mathrm{d}s.\end{aligned} \tag{15}$$

Of course, the expectation in the first line of (15) can be evaluated numerically, but such is very time consuming. Moreover, we show in the next section that the approximation (15) leads to sufficiently accurate results.

For illustration, MATLAB code is given in the appendix, implementing the Brownian bridge scheme. The code implements a single time-step in a single-factor model with constant volatility. These simplications are for clarity of exposition only and are, of course, *not* a restriction imposed by the Brownian bridge scheme.

Brownian Bridge Scheme for Single Time-steps

The performance of the Brownian bridge scheme for single time-steps is investigated both theoretically and numerically.

Theoretically, the result relates to the squared error of a discretization. Let $\{\bar{\mathbf{f}}\}$ be a single time-step discretization. Then the expected squared error is defined by

$$S^2(\{\bar{\mathbf{f}}\}) = E\left\lfloor ||\bar{\mathbf{f}}(\tau_{m+1}) - \mathbf{f}(\tau_{m+1})||^2 \middle| \bar{\mathbf{f}}(\tau_m) = \mathbf{f}(\tau_m)\right\rfloor. \tag{16}$$

Pietersz et al. (2004) establish the following result, which is based on the least-squares property of the expectation or *projection* operator employed in the definition of the Brownian bridge scheme in (11).

Theorem 1 (Least-squares optimality of the Brownian bridge scheme) *The single time-step discretization* $\{\bar{\mathbf{f}}\}$ *that yields least expected squared error* $(\{\bar{\mathbf{f}}\})$ *for the forward rate process (1) over all single time-step discretizations of the form (3) is given by the Brownian bridge scheme (11).*

Numerically, the discretizations of the previous section are compared in the LIBOR-in-arrears test of Hunter et al. (2001). For details of this test, the reader is referred there. Only the idea of the test is briefly described here. Consider a forward rate under the measure of a discount bond maturing at its *fixing* time. The associated forward rate process is not a martingale under this measure, and the respective SDE thus features a stochastic drift term. Nonetheless, an analytical formula for the probability density of the forward rate at its fixing time can be derived. Also, a single time-step discretization implies a certain such density. This discretization-implied density may then be compared to the true density to determine the accuracy of the respective discretization. The results of this test have been displayed in Figure 1. In the left and right panels, the density and error in the density have been depicted, respectively. Parameters are: the 3-months forward rate maturing 30 years from today, with an initial level of 8% and volatility of 24%. The 'BB alternative' scheme indicates full numerical integration of the expectation in Step 4 of the approximation of (11), instead of inserting the mean of the Brownian bridge. Note thus that there is virtually no difference of inserting the mean versus a full numerical integration, establishing the validity of the approximation (15) in Step 4. As can be seen from Figure 1, the Brownian bridge scheme has almost zero discretization bias compared to the other discretizations. The Brownian bridge is thus superior for single time-steps, which can be expected from Theorem 1. Also, Milstein is the worst performing scheme. As mentioned before, this is due to the large time-step and the Milstein-specific quadratic error in the time-step. The numerical results on the Milstein scheme show that we cannot hope to obtain better results with higher-order schemes for single-time steps. Consequently, the Brownian bridge scheme is thus the best one can possibly hope to obtain for single time-steps, over all discretizations (including higher-order ones).

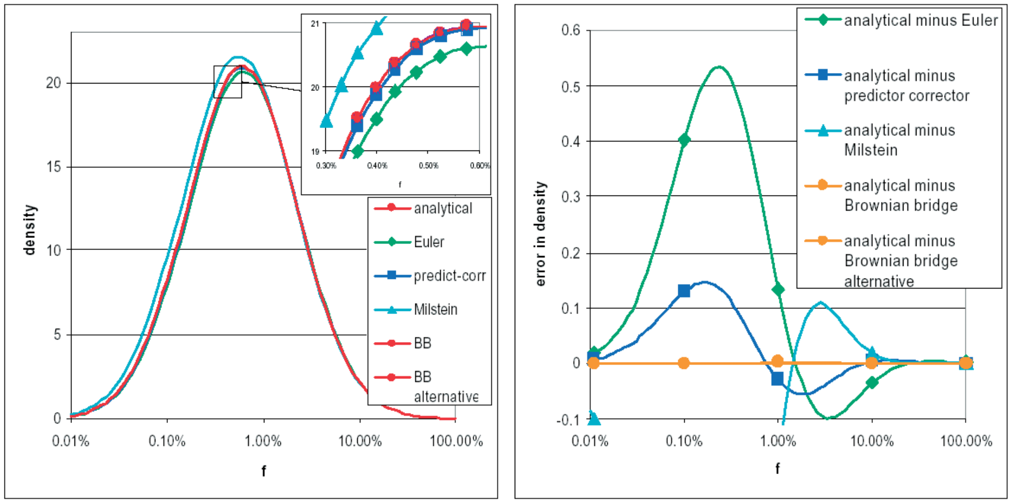

Figure 1: Results of the LIBOR-in-arrears test extended from Hunter et al. (2001)

Brownian Bridge Scheme for Multi Time-steps

In this section, we study the Brownian bridge scheme viewed as a multi time-step MC simulation scheme, both theoretically and numerically, even though the Brownian bridge scheme has been designed for single-time steps. The theoretical result relates to the *weak convergence* of the scheme and the associated *order of convergence*. The maximum step size of a discretization is defined as $\max\{\tau_{m+1} - \tau_m;\, m = 0, \ldots, M\}$.

Definition 1 (Weak convergence) *A scheme $\{\bar{\mathbf{f}}\}$ with maximum step size ε is said to converge weakly to $\{\mathbf{f}\}$ with order β, if for all functions g with $2(\beta + 1)$-polynomially bounded derivatives, there exists a constant c such that for sufficiently small ε,*

$$|E[g(\mathbf{f})] - E[g(\mathbf{f})]| \leq c \times \varepsilon^{\beta}. \tag{17}$$

In finance, we are interested in the correct estimation of prices of derivatives, which are expectations under the risk-neutral measure of derivative-payoffs. Therefore weak convergence is a criterion of financial interest. Pietersz et al. (2004) prove the following result.

Theorem 2 (Weak convergence of the Brownian bridge scheme) *The Brownian bridge scheme (11) converges weakly with order 1 to the forward rates process (1).*

For comparison purposes, the Euler, predictor-corrector and Milstein schemes also converge weakly with orders 1, 1, and 2, respectively. In the remainder of this section, the various discretization schemes are compared numerically. To this order, the discretization biases of a floating leg and of a cap are estimated by MC simulation. To obtain a bias-estimate with minimal standard error, we jointly simulate the values of individual payments in the floating leg and cap under their respective forward measures. Such procedure filters out the discretization bias from the random noise in the simulation. Note that, under the forward measure, there is no drift term and therefore the associated payoff is an unbiased estimator of the value of the contract. If we denote by π_{terminal} and π_{fwd} the numeraire-deflated contract payoff in the terminal and forward measure, respectively, then an unbiased estimator of the bias is $\pi_{\text{terminal}} - \pi_{\text{fwd}}$. Alternatively, we can benchmark against the analytical value of the floating leg or cap, which yields the unbiased estimator of the bias $\pi_{\text{terminal}} - \pi_{\text{analytical}}$. The variances of the two estimators are

$$\begin{aligned} \text{var}[\pi_{\text{terminal}} - \pi_{\text{fwd}}] &= \text{var}[\pi_{\text{terminal}}] + \text{var}[\pi_{\text{fwd}}] \\ &\quad - 2\text{cov}[\pi_{\text{terminal}}, \pi_{\text{fwd}}] \end{aligned}, \tag{18}$$

$$\text{var}[\pi_{\text{terminal}} - \pi_{\text{analytical}}] = \text{var}[\pi_{\text{terminal}}].$$

If we assume $\mathrm{var}[\pi_{\mathrm{terminal}}] \approx \mathrm{var}[\pi_{\mathrm{fwd}}]$, then (18) becomes

$$\mathrm{var}[\pi_{\mathrm{terminal}} - \pi_{\mathrm{fwd}}] \approx \mathrm{var}[\pi_{\mathrm{terminal}}] \times 2 \times (1 - \rho[\pi_{\mathrm{terminal}}, \pi_{\mathrm{fwd}}]) \qquad (20)$$

Therefore, if $\rho[\pi_{\mathrm{terminal}}, \pi_{\mathrm{fwd}}] > \frac{1}{2}$, we have variance reduction. In our numerical LIBOR tests we found $\rho \approx 0.999$, which means that the variance is reduced by a factor of 500. The benchmark against the forward measure payoff is thus also a useful tool when validating an implementation of a LIBOR market model, since a bias that stems from an implementation error is more easily filtered out from the random noise of the MC simulation.

The results of the MC convergence tests have been displayed in Figure 2. The settings of the test are as follows. We use 10,000,000 simulation paths, forward rates at 3%, volatility at 30%, and a 1-factor model. For the cap, we consider a 5-years deal, paying LIBOR over the strike of 2.5% (if at all) fixed at 1, 2, . . ., 5 years and paid at 2, 3, . . ., 6 years. For the floating leg, LIBOR is fixed at 0.25, 0.5, . . ., 1.25 years and paid at 0.5, 0.75, . . ., 1.5 years. The values of the cap and floating leg are 0.0368 and 0.0364, respectively. The results of the test clearly show that any discretization smarter than Euler attains a discretization bias that is statistically indiscernible from the bias obtained by the best performing discretization. In terms of computational time, the ranking from fastest to slowest, in our numerical tests, is: 1. Euler, 2. Milstein, 3. predictor-corrector, 4. Brownian bridge. We emphasize here again that the strength of the Brownian bridge scheme lies in single time-steps, and not in multi time-steps.

Illustration with a Two-factor Model

The purpose of this example is to show that the single time-step pricing framework estimates prices sufficiently accurate for shorter maturity deals, while providing a significant reduction of computational time over Longstaff & Schwartz (2001) American option Monte Carlo pricing. We consider the pricing of a Bermudan swaption in a 2-factor model given by (under the forward measure)

$$\frac{df_i}{f_i} = (18\%) \times \left\{ v_i \, dw_1 + \sqrt{1 - v_i^2} \, dw_2 \right\}, \; v_i = \frac{t_i - t_1}{t_n - t_1}. \qquad (21)$$

This instantaneous volatility form is entirely hypothetical, used for purpose of illustration only. The volatility structure has the property of declining correlation as the expiry difference between forward rates increases. Further parameters: Initial forward rates at 3%, a pay fixed Bermudan swaption with the strike at 3%, we use 50,000 paths in the simulation, and regression on all forward rates available in the model, with a constant term, and one linear term for each forward rate. For the numerical implementation of the two-dimensional PDE, we use the Hopscotch scheme[3], see Wilmott (1998, Paragraph 48.5). The results have been displayed in Table 1. There, *NPV*, *Comp. time*, *Std. err.*, *bp*, and *s*, abbreviate *net present value*, *computational time*, *standard error*, *basis points* and *seconds*, respectively. The results show that the single time-step pricing framework prices the Bermudan swaptions sufficiently accurately, while providing a significant reduction of computational time compared to the least squares Monte Carlo algorithm of Longstaff & Schwartz (2001).

For more extensive numerical tests with the single-time step pricing framework, the reader is referred to Pietersz et al. (2004). The test results reported there include investigation of 1-factor model prices, exercise boundaries and risk sensitivities.

TABLE 1: RESULTS OF THE 2-FACTOR MODEL COMPARISON.

	Brownian bridge		Longstaff & Schwartz (2001)		
Bermudan	Estimated NPV (bp)	Comp. time (s)	Estimated NPV (bp)	Comp time (s)	Std. err. (bp)
2NC1	23.06	0.1	22.59	1	0.15
3NC1	49.92	0.3	49.26	3	0.29
4NC1	78.03	1	77.08	5	0.42
5NC1	105.82	2	105.23	9	0.54
6NC1	133.23	4	130.74	14	0.66
7NC1	166.93	7	165.19	22	0.84
8NC1	200.13	11	197.44	32	0.98
9NC1	230.50	17	228.48	41	1.15

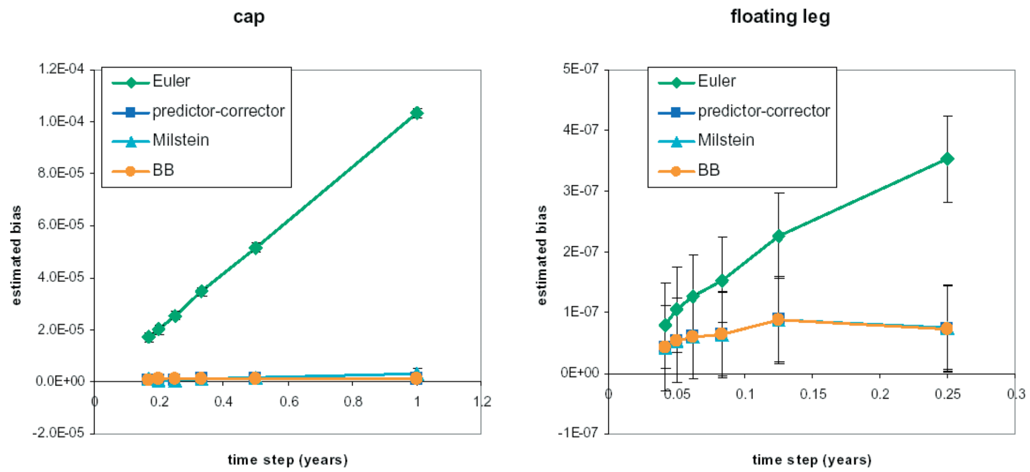

Figure 2: Results of the MC convergence tests

Conclusions

We presented the new Brownian bridge scheme for the LIBOR market model. The Brownian bridge scheme enjoys superior accuracy for single time-steps and is thus particularly apt for use in the single time-step approximate pricing framework.

Appendix: MATLAB Code Illustrating the Brownian Bridge Scheme

```
function result = fBB(n,f0,a,vol,t,z)
% calculates forward LIBOR rates in one-factor model with
% Brownian bridge drift approximation & single time step,
% given the normal increment z

% n, no. of forward LIBORs, a positive integer
% f0, array with n elements, time zero forward LIBORs
% a, array with n elements, day count fractions
% vol, array with n elements, vol[i] = volatility of for-
ward
%  LIBOR i
% t, time (scalar)
% z, Gaussian increment ~N(0,1), scalar

f=zeros(n,1); % creates zero array with n entries, f is
used
              % to store result

% first do ultimate forward LIBOR => martingale!
f(n) =f0(n)*exp(-0.5*vol(n)^ 2*t+vol(n)*sqrt(t)*z);

% loop from penultimate LIBOR down to first LIBOR
run_drift =0.0; % used for efficient calculation of drift
for i =n-1:-1:1
    zt =log(f(i+1)/f0(i+1))+0.5*vol(i+1)^ 2*t; % needed
for
        % 'driftBB' function below
    % quad is a standard integration routine in MATLAB
    % quad(@f,a,b,tol,trace,p1,p2,...) integrates the
    % function f(s,p1,p2,...) over s from a to b with
    % convergence criteria tol and trace
    % for definitions of tol and trace we refer to MATLAB
    % documentation
    % of course, one can use any other integration routine
    % instead of quad
    % adjusting the convergence criterion of the numerical
    % integrator allows for a trade-off between accuracy
    % and computational speed
    % for example, the predictor-corrector scheme is a
    % special case of the Brownian bridge scheme if the
    % crudest integrator (two-point trapezoid) is used
    run_drift = run_drift-quad(@driftBB,0.0,t,1.0e-
      6,0,f0(i+1),a(i+1),vol(i+1),t,zt);
    f(i) =f0(i)*exp((run_drift*vol(i)-0.5*vol(i)^
      2*t)+vol(i)*sqrt(t)*z); % Eqtn (4)
end

result = f; % return result f

function result = driftBB(s,f0,a,vol,t,zt)
% calculates drift term evaluated at the mean of the
Brownian
% bridge. this function will be integrated over time.

% s, scalar, current (intermediate) time
% f0, scalar, time zero forward LIBOR
% a, scalar, day count fraction
% vol, scalar, volatility of forward LIBOR
% t, scalar, time (at which forward LIBOR has already been
%   predicted)
% zt, scalar, help variable associated with forward LIBOR
%   predicted at time t

% mean of Brownian bridge
m=s./t.*zt-0.5.*vol.^2.*s.*s./t+log(f0)+log(a); % Eqtn
(14)
%   in log-form
result =vol*exp(m)./(1.0+exp(m)); % the essential form of
the
%   BGM drift in terms of log rates:exp(.)/(1+exp(.))
```

FOOTNOTES & REFERENCES

1. The reason for working with log rates, instead of, for example, rates or (log) relative discount bond prices, is that log rates obtain the smallest simulation discretization error, by far. See Glasserman & Merener (2003, Section 7).
2. The reason for working with the terminal measure is that the time-t terminal numeraire b_{n+1} is fully determined by time-t forward rates, since $b_{n+1}(t_i) = 1/\prod_{j=i}^{n}(1+\delta_j f_j(t_i))$. This implies that, under the terminal measure, the numeraire-deflated payoff of an interest rate derivative can be fully determined by time-t forward rates, and thus we can estimate prices on a finite difference grid. Such can never be the case for the spot numeraire, since $b_{\text{spot}}(t_i) = \prod_{j=0}^{i-1}(1+\delta_j f_j(t_j))$, which is clearly path-dependent.
3. We thank Glyn Baker for the use of his Hopscotch implementation.

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